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NATIONAL ADVISORY COMMITTEE
FOR AERONAUTICS



● TECHNICAL MEMORANDUM 1263

CONTRIBUTION TO THE PROBLEM OF BUCKLING OF ORTHOTROPIC
PLATES, WITH SPECIAL REFERENCE TO PLYWOOD

By Wilhelm Thielemann

Translation of "Beitrag zur Frage der Beulung orthotroper Platten,
insbesondere von Sperrholzplatten"



Washington
August 1950

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SUMMARY

The first part of the present report deals with the theory of elasticity of orthotropic plates. The general differential equation of the bending surface of orthotropic rectangular plates whose principal directions of stiffness are not parallel to the plate edges (general-orthotropic plate) is indicated. The possibility of applying the theory of elasticity of orthotropic plates to plywood is investigated and the relationship between the elastic moduli and the angle between the load direction and the principal stiffness directions is described for plywood of various constructions.

The second part is concerned with the stability equation of the general-orthotropic rectangular plate under uniform shear and axial loads. This differential equation, which, compared to the differential equation of the orthotropic rectangular plate whose principal directions of stiffness are parallel to the plate edges, contains additional terms, can be solved for a very long plate strip by the Southwell-Skan formula for the isotropic plate strip. Besides the exact solution an approximate solution is given. For the most important practical case of orthotropic plate strips, whose principal directions of stiffness are inclined at 45° with respect to the plate edges, the buckling loads and buckling lengths in pure compression and pure shear are plotted against the stiffness values of the plate. For the special stiffness values of plywood plates of various constructions, the buckling loads and buckling wave lengths were determined for all inclinations of principal stiffness directions (0° to 180°) relative to plate edges and also plotted for pure compressive and pure shear loading.

*"Beitrag zur Frage der Beulung orthotroper Platten, insbesondere von Sperrholzplatten."

I. I N T R O D U C T I O N

Thin plywood strips are frequently used in airplane designs as spar webs, wing covering, etc. They are subjected to forces within the total assembly of the structural member which usually lie in its plane and stress the plates in compression or shear. Since the plates are frequently cut so thin that buckling under such loads is to be expected, the knowledge of the critical loads at which the plates change from their originally straight, stable position into a new, buckled position should be of interest.

Being built up of plies at right angles to each other, the plywood plates are regarded as orthogonally-anisotropic plates (orthotropic plates according to the term introduced by Huber). By its construction the stiffness of the plate is dependent on the position of the elastic axis relative to the direction of the fiber (grain).

Although the plywood plate cannot be termed completely homogeneous because of the violent property fluctuations over the wall thickness, the subsequent investigations of the buckling stiffness of orthotropic plates nevertheless are generally made for a homogeneous orthotropic and elastic material. The effect of the incomplete homogeneity of the plywood plate on the stiffness, etc., is particularly investigated.

The problem of stability of orthotropic plates has been treated by C. Schmieden (reference 1), Bergmann-Reissner (reference 2), and Seydel (reference 3), with special reference to the case of pure shear loading in connection with the Southwell-Skan investigations on the stability of isotropic plates (reference 4). Basic investigations on the orthotropic plate have been made by Huber (reference 5) who also set up the differential equation of bending of the orthotropic plate. Further data on the stability of orthotropic plates, including the data on buckling load under compressive stress, are given by Timoshenko (reference 6).

But all these reports deal only with orthotropic plates with principal directions of stiffness (on plywood, the directions of the grain of the inner and outer ply) parallel to the edges of the rectangular plate (termed "special orthotropic plate" hereinafter). For the use of orthotropic plates, especially of plywood, as spar or wing covering, however, the knowledge of the buckling strength of plates with principal stiffness other than parallel to the edges is of interest.

In spars with plywood webs the direction of the fibers is usually at 45° to the flanges, since under shearing stress - apart from an increase in shear stiffness - a substantial increase in buckling load and hence often also in the ultimate load of the web relative to a plate fixed parallel can be obtained.

In order to include these important practical cases also, the problem of buckling of special-orthotropic plates, so exhaustively explored by Seydel for shear load, is extended to include the problem of buckling of orthotropic plates with arbitrary principal stiffnesses relative to the plate edges (termed "general-orthotropic plate" hereinafter).

II. THEORY OF ELASTICITY OF THE GENERAL-ORTHOTROPIC PLATE

The investigation of buckling of the general-orthotropic plate requires a closer insight into the elasticity theory of such plates. The general-orthotropic plate is characterized by the inclination of the principal stiffness directions (of the principal axes system) of the plate with respect to the coordinate system to which the stresses and strains are referred. For the special-orthotropic plate, the principal system of axes is coincident with the coordinate system (figs. 1(a) and 1(b)). The inclination of the principal system of axes with respect to the system of coordinates of the plate necessitates a construction of the laws of elongation of the general-orthotropic plate which, compared to the strain laws of the special-orthotropic plate, contains additional terms. Whereas the elastic properties of the special-orthotropic plate can be defined by four elasticity constants, the identification of the properties of the general-orthotropic plate requires the knowledge of six constants. Hence, aside from the stress-strain law - as will be shown in the subsequent sections - the differential equation of the bending surface, and the equation of energy of the general-orthotropic plate for the bending of these plates (both equations being used for the further investigation), two additional terms are obtained which contain these new additive elastic constants.

1. The General Stress-Strain Law of the Orthotropic

Plate (Plane Stress Condition)

(a) The Elongation Equations

The law of elongation of the general-orthotropic plate can be written in the form (reference 5)

$$\left. \begin{aligned} \epsilon_x &= \alpha_{11}'\sigma_x + \alpha_{12}'\sigma_y + \alpha_{13}'\tau \\ \epsilon_y &= \alpha_{21}'\sigma_x + \alpha_{22}'\sigma_y + \alpha_{23}'\tau \\ \gamma &= \alpha_{31}'\sigma_x + \alpha_{32}'\sigma_y + \alpha_{33}'\tau \end{aligned} \right\} \quad (1)$$

The moduli of elasticity α' for the coordinate system xy are dependent on the angle ω of the system, which the principal axes system forms with the coordinate system. (These general moduli of elasticity are hereafter denoted by α'_{ik} , the special moduli (for the case $\omega = 0$) with α_{ik} .)

It can be proved (reference 5) that the symmetry conditions $\alpha_{12}' = \alpha_{21}'$, $\alpha_{13}' = \alpha_{31}'$, and $\alpha_{23}' = \alpha_{32}'$ exist, hence that the elongation law of the general-orthotropic plate is represented by six elastic constants.

The moduli α_{13}' and α_{23}' in equation (1) are a measure for the elongations due to shear stresses in the general case ($\omega \neq 0$) and the displacements induced by axial stresses, respectively.

These moduli α_{13}' and α_{23}' disappear in the case of the special orthotropic plate ($\omega = 0$) as will be shown later, so that this case is characterized by four constants.

Introduction of the conventional reciprocal expressions for the moduli of elasticity

$$\alpha_{11}' = \frac{1}{E_{11}}; \alpha_{22}' = \frac{1}{E_{22}}; \alpha_{12}' = \alpha_{21}' = -\frac{\nu_{11}}{E_{11}} = -\frac{\nu_{22}}{E_{22}}; \alpha_{33}' = \frac{1}{G}$$

gives for the case of special orthotropy the equations:

$$\left. \begin{aligned} \epsilon_x &= \alpha_{11}'\sigma_x + \alpha_{12}'\sigma_y = \frac{1}{E_{11}}(\sigma_x - \nu_{11}\sigma_y) \\ \epsilon_y &= \alpha_{12}'\sigma_x + \alpha_{22}'\sigma_y = \frac{1}{E_{22}}(\sigma_y - \nu_{22}\sigma_x) \\ \gamma &= \alpha_{33}'\tau = \frac{\tau}{G} \end{aligned} \right\} \quad (1a)$$

On top of that, $\alpha_{11}' = \alpha_{22}' = \frac{1}{E}$, for the case of the isotropic plate, so that the equations of elongation read:

$$\left. \begin{aligned} \epsilon_x &= \alpha_{11}'\sigma_x + \alpha_{12}'\sigma_y = \frac{1}{E}(\sigma_x - \nu\sigma_y) \\ \epsilon_y &= \alpha_{12}'\sigma_x + \alpha_{11}'\sigma_y = \frac{1}{E}(\sigma_y - \nu\sigma_x) \\ \gamma &= \alpha_{33}'\tau = \frac{\tau}{G} \end{aligned} \right\} \quad (1b)$$

And since a further relation can be established between the moduli α_{11} , α_{12} , and α_{33} , the elastic constants for isotropic material are reduced to two.

(b) Determination of Moduli of Elasticity

For the general-orthotropic plate, the elastic constants in equation (1) are dependent on the angle ω formed by the principal axes of the orthotropic plate with the coordinate system. In the following, the moduli of elasticity for any system of coordinates orientated with respect to the principal axes of the plate are determined in relation to the moduli of elasticity for the principal axes direction and the angle formed by the coordinate axes relative to the principal axes.

From a general-orthotropic plate element of which one principal axis is inclined at angle ω to the x axis, a rectangle is cut out in such a way that the edges of the rectangle lie parallel to the principal axes of the plate element. The plate element is loaded along its edges by the stresses σ_x , σ_y and τ (fig. 2). These stresses in turn introduce stresses σ_1 , σ_2 , and τ_0 at the cut-out rectangle, which can be indicated immediately for the plane stress condition with the aid of the conversion formulas

$$\left. \begin{aligned} \sigma_1 &= \sigma_x \cos^2 \omega + \sigma_y \sin^2 \omega - 2\tau \sin \omega \cos \omega \\ \sigma_2 &= \sigma_x \sin^2 \omega + \sigma_y \cos^2 \omega + 2\tau \sin \omega \cos \omega \\ \tau_0 &= \sigma_x \sin \omega \cos \omega - \sigma_y \sin \omega \cos \omega + \tau(\cos^2 \omega - \sin^2 \omega) \end{aligned} \right\} \quad (2)$$

Denoting the diagonal of the rectangular element parallel to the x axis with l , the sides of the rectangle have the lengths $\sin \omega$ and $\cos \omega$ (fig. 3(a)). Let Δl_1 and Δl_2 be the displacement components of a rectangular point of the element distorted by the stresses σ_1 , σ_2 , and τ_0 along the edges of the rectangle. The displacement components of the rectangular point along the x axis, that is, the elongation of the rectangular diagonal of magnitude l , can be read from figure 3(b) as

$$\epsilon_x = \Delta l_x = \Delta l_1 \cos \omega + \Delta l_2 \sin \omega \quad (3a)$$

The angle of shearing strain through which the connecting line of the distorted rectangular point has turned with respect to its original position - that is, the x axis - is (fig. 3(b)):

$$\gamma = 2[-\Delta l_1 \sin \omega + \Delta l_2 \cos \omega] \quad (3b)^*$$

From a corresponding rectangular element, whose diagonal of length l runs parallel to the y axis, the displacement component of the rectangular point and the elongation along the y axis follow as

$$\epsilon_y = \Delta l_y = \Delta l_1 \sin \omega + \Delta l_2 \cos \omega \quad (3c)$$

The displacement components Δl_1 and Δl_2 along the rectangular sides consist of displacements due to axial strain, transverse contraction, and shearing strain of the rectangular element.

From figures 4(a) to 4(c), where the several contributions of the deformation strain are represented separately, follow immediately:

(a) The displacement due to longitudinal deformation

$$\left. \begin{aligned} \Delta l_1 &= \epsilon_1 \cos \omega = \sigma_1 \alpha_{11} \cos \omega \\ \Delta l_2 &= \epsilon_2 \sin \omega = \sigma_2 \alpha_{22} \sin \omega \end{aligned} \right\} \quad (4a)$$

(b) Displacement due to transverse contraction

$$\left. \begin{aligned} \Delta l_1 &= \epsilon_{Q1} \cos \omega = \sigma_2 \alpha_{12} \cos \omega \\ \Delta l_2 &= \epsilon_{Q2} \sin \omega = \sigma_1 \alpha_{12} \sin \omega \end{aligned} \right\} \quad (4b)$$

(c) Displacement due to shearing deformation

$$\left. \begin{aligned} \Delta l_1 &= \gamma_o \frac{\sin \omega}{2} = \tau_o \alpha_{33} \frac{\sin \omega}{2} \\ \Delta l_2 &= \gamma_o \frac{\cos \omega}{2} = \tau_o \alpha_{33} \frac{\cos \omega}{2} \end{aligned} \right\} \quad (4c)$$

* NACA Reviewer's note: This equation was incorrectly written as $2\gamma = -\Delta l_1 \sin \omega + \Delta l_2 \cos \omega$ in the original German version of this paper.

The expressions (4a) to (4c) introduced in equation (3a) give the elongation ϵ_x

$$\epsilon_x = \Delta l_x = \Delta l_1 \cos \omega + \Delta l_2 \sin \omega$$

$$\begin{aligned} \epsilon_x = & \sigma_1 \alpha_{11} \cos^2 \omega + \sigma_2 \alpha_{12} \cos^2 \omega + \tau_0 \alpha_{33} \frac{\sin \omega \cos \omega}{2} \\ & + \sigma_2 \alpha_{22} \sin^2 \omega + \sigma_1 \alpha_{12} \sin^2 \omega + \tau_0 \alpha_{33} \frac{\sin \omega \cos \omega}{2} \end{aligned}$$

The stresses σ_1 , σ_2 , and τ_0 can be replaced by means of the equations (2) by the stresses σ_x , σ_y and τ , so that the final expression for the elongation ϵ_x of the general-orthotropic plate reads

$$\begin{aligned} \epsilon_x = & \sigma_x [\alpha_{11} \cos^4 \omega + (2\alpha_{12} + \alpha_{33}) \sin^2 \omega \cos^2 \omega + \alpha_{22} \sin^4 \omega] \\ & + \sigma_y [(\alpha_{11} + \alpha_{22}) \cos^2 \omega \sin^2 \omega + \alpha_{12} (\cos^4 \omega + \sin^4 \omega) - \alpha_{33} \cos^2 \omega \sin^2 \omega] \\ & + \tau [-2(\alpha_{11} \cos^2 \omega - \alpha_{22} \sin^2 \omega) \sin \omega \cos \omega \\ & + (2\alpha_{12} + \alpha_{33}) \sin \omega \cos \omega (\cos^2 \omega - \sin^2 \omega)] \end{aligned} \quad (5a)$$

The shearing strain of the general-orthotropic plate follows from equations (3b) and (4a) to (4c) as

$$\begin{aligned} \gamma = & \sigma_x [-2(\alpha_{11} \cos^2 \omega - \alpha_{22} \sin^2 \omega) \sin \omega \cos \omega \\ & + (2\alpha_{12} + \alpha_{33}) \sin \omega \cos \omega (\cos^2 \omega - \sin^2 \omega)] \\ & + \sigma_y [-2(\alpha_{11} \sin^2 \omega - \alpha_{22} \cos^2 \omega) \sin \omega \cos \omega \\ & - (2\alpha_{12} + \alpha_{33}) \sin \omega \cos \omega (\cos^2 \omega - \sin^2 \omega)] \\ & + \tau [4(\alpha_{11} + \alpha_{22} - 2\alpha_{12}) \cos^2 \omega \sin^2 \omega + \alpha_{33} (\cos^2 \omega - \sin^2 \omega)^2] \end{aligned} \quad (5b)$$

The elongation ϵ_y can be found by corresponding considerations from equation (3c) as

$$\begin{aligned} \epsilon_y = & \sigma_x \left[(\alpha_{11} + \alpha_{22}) \cos^2 \omega \sin^2 \omega + \alpha_{12} (\cos^4 \omega + \sin^4 \omega) - \alpha_{33} \sin^2 \omega \cos^2 \omega \right] \\ & + \sigma_y \left[\alpha_{11} \sin^4 \omega + (2\alpha_{12} + \alpha_{33}) \sin^2 \omega \cos^2 \omega + \alpha_{22} \cos^4 \omega \right] \\ & + \tau \left[-2(\alpha_{11} \sin^2 \omega - \alpha_{22} \cos^2 \omega) \sin \omega \cos \omega \right. \\ & \left. - (2\alpha_{12} + \alpha_{33}) \cos \omega \sin \omega (\cos^2 \omega - \sin^2 \omega) \right] \end{aligned} \quad (5c)$$

Comparison of the coefficients of equation (1) and (5) yields for the moduli of elasticity of a coordinate system inclined at the angle ω relative to one principal axis, the following relations

$$\left. \begin{aligned} \alpha_{11}' &= \alpha_{11} \cos^4 \omega + (2\alpha_{12} + \alpha_{33}) \sin^2 \omega \cos^2 \omega + \alpha_{22} \sin^4 \omega \\ \alpha_{22}' &= \alpha_{11} \sin^4 \omega + (2\alpha_{12} + \alpha_{33}) \sin^2 \omega \cos^2 \omega + \alpha_{22} \cos^4 \omega \\ \alpha_{12}' &= (\alpha_{11} + \alpha_{22}) \cos^2 \omega \sin^2 \omega + \alpha_{12} (\cos^4 \omega + \sin^4 \omega) - \alpha_{33} \sin^2 \omega \cos^2 \omega \\ \alpha_{33}' &= 4(\alpha_{11} + \alpha_{22} - 2\alpha_{12}) \cos^2 \omega \sin^2 \omega + \alpha_{33} (\cos^2 \omega - \sin^2 \omega) \\ \alpha_{13}' &= -2(\alpha_{11} \cos^2 \omega - \alpha_{22} \sin^2 \omega) \sin \omega \cos \omega \\ &\quad + (2\alpha_{12} + \alpha_{33}) \sin \omega \cos \omega (\cos^2 \omega - \sin^2 \omega) \\ \alpha_{23}' &= -2(\alpha_{11} \sin^2 \omega - \alpha_{22} \cos^2 \omega) \sin \omega \cos \omega \\ &\quad - (2\alpha_{12} + \alpha_{33}) \cos \omega \sin \omega (\cos^2 \omega - \sin^2 \omega) \end{aligned} \right\} \quad (6)$$

As essential result of the determination of the moduli of elasticity, it is found that the six constants of the general-orthotropic plate can be derived from the four constants of the special-orthotropic plate (principal direction constants α_{11} , α_{22} , α_{12} , and α_{33}), so that additional experimental determinations of moduli on plates beyond these four quantities are not necessary.

For isotropic material, the conventional relation between the modulus of rigidity, Young's modulus, and transverse contraction exists

$$G = \frac{E}{2(1 + \nu)} \quad (7)$$

It is logical to expect a corresponding relation for orthotropic material, so that here also the experimental determination of the modulus of rigidity would be superfluous.

The modulus of rigidity of a general-orthotropic plate is according to equation (6)

$$\alpha_{33}' = 4(\alpha_{11} + \alpha_{22} - 2\alpha_{12}) \sin^2\omega \cos^2\omega + \alpha_{33} (\cos^2\omega - \sin^2\omega)^2 \quad (6a)$$

This relation indicates that the modulus of rigidity α_{33}' is independent of the modulus α_{33} valid for the principal direction of the orthotropic plate only when $\omega = 45^\circ$ (hence, when the directions of the principal axes of the plate coincide with the principal stress directions in pure shear) and that the dependence is limited to the moduli of elasticity and transverse contractions in the principal directions. Hence, in this case

$$\alpha_{33}' = \alpha_{11} + \alpha_{22} - 2\alpha_{12} \quad (8)$$

Written in the conventional manner, equation (8) reads:

$$G' = \frac{1}{\frac{1}{E_{11}} + \frac{1}{E_{22}} + \frac{\nu_{11}}{E_{11}} + \frac{\nu_{22}}{E_{22}}} = \frac{E_{11}E_{22}}{E_{11}(1 + \nu_{22}) + E_{22}(1 + \nu_{11})} \quad (8a)$$

Hertel (reference 7, p. 135) also derived this equation for this particular case ($\omega = 45^\circ$) for the shear modulus of plywood (reference 7). In all other cases ($\omega \neq 45^\circ$) the determination of shear modulus α_{33}' is predicated on the knowledge of shear modulus α_{33} for the principal axis ($\omega = 0$). Even in the case $\omega = 45^\circ$ the experimental determination of α_{33} cannot be dispensed with, since modulus α_{33} must be known in order to determine the moduli of elasticity α_{11}' , α_{22}' , and α_{12}' , in this direction, according to equation (6).

So, in contrast to the isotropic plate, the experimental determination of the shear modulus α_{33} independently of all other principal elasticity moduli is necessary in every case.

In this connection, another particular case of the general-orthotropic plate is pointed out:

If, as may happen in measurements, the relation

$$\alpha_{11} + \alpha_{22} - 2\alpha_{12} = \alpha_{33}$$

exists between the moduli of the principal axes, the moduli of the plate in the remaining directions ($\omega \neq 0$) are

$$\left. \begin{aligned} \alpha_{12}' &= \alpha_{12} = \text{Constant} \\ \alpha_{33}' &= \alpha_{33} = \text{Constant} \\ \alpha_{11}' &= \alpha_{11} \cos^4 \omega + (\alpha_{11} + \alpha_{22}) \sin^2 \omega \cos^2 \omega + \alpha_{22} \sin^4 \omega \\ \alpha_{22}' &= \alpha_{11} \sin^4 \omega + (\alpha_{11} + \alpha_{22}) \sin^2 \omega \cos^2 \omega + \alpha_{22} \cos^4 \omega \\ \alpha_{11}' + \alpha_{22}' &= \alpha_{11} + \alpha_{22} = \text{Constant} \\ \alpha_{13}' &= \alpha_{23}' = (\alpha_{22} - \alpha_{11}) \sin \omega \cos \omega \end{aligned} \right\} \quad (10)$$

as is readily apparent from equations (6).

Thus, in this particular case, the modulus of shear and the modulus of transverse contraction are independent of the angle of direction, while moduli α_{11}' , α_{22}' , α_{13}' , and α_{23}' continue to be dependent on ω .

A particular case of this kind, in which the general-orthotropic plate has isotropic properties with respect to α_{33}' and α_{12}' , is represented in close approximation by a three-ply beech plywood plate. (Compare Section II 5, fig. 18(a).)

For completeness the derivation of (7) governing isotropic material from equations (6) and (10) is included.

For isotropic material all moduli must be independent of ω , that is, be constant in every direction. The requirement is met, according to equations (10) when the following relations exist:

$$\begin{aligned}\alpha_{11} &= \alpha_{22} \\ \alpha_{11} + \alpha_{22} - 2\alpha_{12} &= \alpha_{33}\end{aligned}\tag{11}$$

Utilizing the first equation of (11), the second takes the form

$$2(\alpha_{11} - \alpha_{12}) = \alpha_{33}\tag{12}$$

and this equation corresponds, in different form, exactly to the relation (7).

Equations (7) show further that the moduli of elasticity α_{13}' and α_{23}' disappear for $\omega = 0^\circ$ and 90° , that is, for the principal axes directions. They also disappear for the particular case $\omega = 45^\circ$, when the principal moduli of elasticity α_{11} and α_{22} are of equal magnitude.

It is to be noted further that in the case $\omega = 45^\circ$ the relations

$$\alpha_{11}' = \alpha_{22}' \quad \text{and} \quad \alpha_{13}' = \alpha_{23}'$$

are valid.

(c) The Stress Equations

To continue the study of the orthotropic plate, the presentation of the stress condition in the plate in relation to the strain condition is required. These relations can be obtained from equation (1) by solution with respect to the stress components. The ensuing equations have the form

$$\begin{aligned}\sigma_x &= a_{11}'\epsilon_x + a_{12}'\epsilon_y + a_{13}'\gamma \\ \sigma_y &= a_{21}'\epsilon_x + a_{22}'\epsilon_y + a_{23}'\gamma \\ \tau &= a_{31}'\epsilon_x + a_{32}'\epsilon_y + a_{33}'\gamma\end{aligned}\tag{13}$$

The connections between the elasticity moduli α' and the elastic parameters a' may be obtained according to the rules of solving equation systems.

Accordingly

$$\sigma_x = \frac{\Delta_1}{\Delta}; \quad \sigma_y = \frac{\Delta_2}{\Delta}; \quad \tau = \frac{\Delta_3}{\Delta} \quad (14)$$

where Δ represents the determinants of the system of equations (1):

$$\Delta = \begin{vmatrix} \alpha_{11}' & \alpha_{12}' & \alpha_{13}' \\ \alpha_{21}' & \alpha_{22}' & \alpha_{23}' \\ \alpha_{31}' & \alpha_{32}' & \alpha_{33}' \end{vmatrix} \quad (15)$$

while the quantities Δ_1 , Δ_2 , and Δ_3 are indicated by the following determinants:

$$\Delta_1 = \begin{vmatrix} \epsilon_x & \alpha_{12}' & \alpha_{13}' \\ \epsilon_y & \alpha_{22}' & \alpha_{23}' \\ \gamma & \alpha_{32}' & \alpha_{33}' \end{vmatrix} \quad \Delta_2 = \begin{vmatrix} \alpha_{11}' & \epsilon_x & \alpha_{13}' \\ \alpha_{21}' & \epsilon_y & \alpha_{23}' \\ \alpha_{31}' & \gamma & \alpha_{33}' \end{vmatrix} \quad \Delta_3 = \begin{vmatrix} \alpha_{11}' & \alpha_{12}' & \epsilon_x \\ \alpha_{21}' & \alpha_{22}' & \epsilon_y \\ \alpha_{31}' & \alpha_{32}' & \gamma \end{vmatrix} \quad (16)$$

The solution of the determinants (16), while making use of the equations (14), gives

$$\begin{aligned} \Delta \sigma_x &= (\alpha_{22}' \alpha_{33}' - \alpha_{23}'^2) \epsilon_x + (-\alpha_{12}' \alpha_{33}' + \alpha_{13}' \alpha_{23}') \epsilon_y \\ &\quad + (\alpha_{12}' \alpha_{23}' - \alpha_{22}' \alpha_{13}') \gamma \\ \Delta \sigma_y &= (-\alpha_{12}' \alpha_{33}' + \alpha_{13}' \alpha_{23}') \epsilon_x + (\alpha_{11}' \alpha_{33}' - \alpha_{13}'^2) \epsilon_y \\ &\quad + (\alpha_{12}' \alpha_{13}' - \alpha_{11}' \alpha_{23}') \gamma \\ \Delta \tau &= (\alpha_{12}' \alpha_{23}' - \alpha_{22}' \alpha_{13}') \epsilon_x + (\alpha_{12}' \alpha_{13}' - \alpha_{11}' \alpha_{23}') \epsilon_y \\ &\quad + (\alpha_{11}' \alpha_{22}' - \alpha_{12}'^2) \gamma \end{aligned} \quad (17)$$

The solution of the determinants gives

$$\Delta = \alpha_{11}'\alpha_{22}'\alpha_{33}' - (\alpha_{11}'\alpha_{23}'^2 + \alpha_{22}'\alpha_{13}'^2 + \alpha_{33}'\alpha_{12}'^2) + 2\alpha_{12}'\alpha_{13}'\alpha_{23}' \quad (15a)$$

As can be proved after considerable calculation by introduction of (6) in (15a), quantity Δ - for an orthotropic plate with properties defined by the four principal constants - is an invariant, which is independent of the position of the coordinate system relative to the principal axes, and which gives

$$\Delta = \alpha_{33}'(\alpha_{11}'\alpha_{22}' - \alpha_{12}'^2) \quad (15b)$$

Comparison of the coefficients between the equations (16) and (17) gives the connections between the elasticity moduli α' and the elastic parameters a' :

$$\begin{aligned} \Delta a_{11}' &= \alpha_{22}'\alpha_{33}' - \alpha_{23}'^2 \\ \Delta a_{12}' &= \Delta a_{21}' = -\alpha_{12}'\alpha_{33}' + \alpha_{13}'\alpha_{23}' \\ \Delta a_{22}' &= \alpha_{11}'\alpha_{33}' - \alpha_{13}'^2 \\ \Delta a_{33}' &= \alpha_{11}'\alpha_{22}' - \alpha_{12}'^2 \\ \Delta a_{13}' &= \Delta a_{31}' = \alpha_{12}'\alpha_{23}' - \alpha_{22}'\alpha_{13}' \\ \Delta a_{23}' &= \Delta a_{32}' = \alpha_{12}'\alpha_{13}' - \alpha_{11}'\alpha_{23}' \end{aligned} \quad (18)$$

The equations (18) indicate that the six elastic parameters a' of the general-orthotropic plate, like the elasticity moduli α' , can

be represented with respect to the four elasticity moduli α_{11} , α_{22} , α_{12} , and α_{33} of the special-orthotropic plate in the principal directions, and which gives, after introduction of equations (6) in (18):

$$\begin{aligned}
 a_{11}' &= \frac{4}{\alpha_{33}} \cos^2 \omega \sin^2 \omega + \frac{\alpha_{11} \sin^4 \omega + \alpha_{22} \cos^4 \omega - 2\alpha_{12} \cos^2 \omega \sin^2 \omega}{\alpha_{11}\alpha_{22} - \alpha_{12}^2} \\
 a_{22}' &= \frac{4}{\alpha_{33}} \cos^2 \omega \sin^2 \omega + \frac{\alpha_{11} \cos^4 \omega + \alpha_{22} \sin^4 \omega - 2\alpha_{12} \cos^2 \omega \sin^2 \omega}{\alpha_{11}\alpha_{22} - \alpha_{12}^2} \\
 a_{12}' &= -\frac{4}{\alpha_{33}} \cos^2 \omega \sin^2 \omega + \frac{(\alpha_{11} + \alpha_{22})\cos^2 \omega \sin^2 \omega - \alpha_{12}(\sin^4 \omega + \cos^4 \omega)}{\alpha_{11}\alpha_{22} - \alpha_{12}^2} \\
 a_{33}' &= \frac{(\sin^2 \omega - \cos^2 \omega)^2}{\alpha_{33}} + \frac{(\alpha_{11} + \alpha_{22} + 2\alpha_{12})\cos^2 \omega \sin^2 \omega}{\alpha_{11}\alpha_{22} - \alpha_{12}^2} \\
 * a_{13}' &= -\cos \omega \sin \omega \left[\frac{2}{\alpha_{33}} (\sin^2 \omega - \cos^2 \omega) + \frac{-\alpha_{11} \sin^2 \omega + \alpha_{22} \cos^2 \omega - \alpha_{12} (\sin^2 \omega - \cos^2 \omega)}{\alpha_{11}\alpha_{22} - \alpha_{12}^2} \right] \\
 * a_{23}' &= \cos \omega \sin \omega \left[\frac{2}{\alpha_{33}} (\sin^2 \omega - \cos^2 \omega) + \frac{\alpha_{11} \cos^2 \omega - \alpha_{22} \sin^2 \omega - \alpha_{12} (\sin^2 \omega - \cos^2 \omega)}{\alpha_{11}\alpha_{22} - \alpha_{12}^2} \right]
 \end{aligned}
 \tag{18a}$$

In the case of the special-orthotropic plate, the expressions for a_{13}' and a_{23}' are considerably simplified, since α_{13}' and α_{23}' disappear, and result in the conventional relations already indicated by Huber (reference 5):

* These two equations were incorrect in the original version of the paper and have been corrected by the NACA Reviewer.

$$\begin{aligned}
 a_{11} &= \frac{\alpha_{22}\alpha_{33}}{\Delta} = \frac{\alpha_{22}}{\alpha_{11}\alpha_{22} - \alpha_{12}^2} = \frac{E_{11}}{1 - \nu_{11}\nu_{22}} \\
 a_{22} &= \frac{\alpha_{11}\alpha_{33}}{\Delta} = \frac{\alpha_{11}}{\alpha_{11}\alpha_{22} - \alpha_{12}^2} = \frac{E_{22}}{1 - \nu_{11}\nu_{22}} \\
 a_{12} &= \frac{-\alpha_{12}\alpha_{33}}{\Delta} = \frac{-\alpha_{12}}{\alpha_{11}\alpha_{22} - \alpha_{12}^2} = \frac{\nu_{11}E_{22}}{1 - \nu_{11}\nu_{22}} = \frac{\nu_{22}E_{11}}{1 - \nu_{11}\nu_{22}} \\
 a_{33} &= \frac{\alpha_{11}\alpha_{22} - \alpha_{12}^2}{\Delta} = \frac{1}{\alpha_{33}} = G
 \end{aligned} \tag{19}$$

$$a_{13} = a_{23} = 0$$

The stress equations of the special-orthotropic plate read then:

$$\begin{aligned}
 \sigma_x &= \frac{E_{11}}{1 - \nu_{11}\nu_{22}} \epsilon_x + \frac{\nu_{11}E_{22}}{1 - \nu_{11}\nu_{22}} \epsilon_y \\
 \sigma_y &= \frac{\nu_{22}E_{11}}{1 - \nu_{11}\nu_{22}} \epsilon_x + \frac{E_{22}}{1 - \nu_{11}\nu_{22}} \epsilon_y \\
 \tau &= G\gamma
 \end{aligned} \tag{20}$$

The further simplified constants for isotropic material read:

$$a_{11} = a_{22} = \frac{E}{1 - \nu^2}; \quad a_{12} = \frac{\nu E}{1 - \nu^2}; \quad a_{33} = G \tag{21}$$

and the stress equations take the conventional form

$$\begin{aligned}
 \sigma_x &= \frac{E}{1 - \nu^2} (\epsilon_x + \nu \epsilon_y) \\
 \sigma_y &= \frac{E}{1 - \nu^2} (\nu \epsilon_x + \epsilon_y) \\
 \tau &= G\gamma
 \end{aligned} \tag{22}$$

For the subsequent applications, the elastic parameters of a general-orthotropic plate for the case $\omega = 45^\circ$ are also of interest.

Equations (18a) give for this case:

$$\begin{aligned}
 a_{11}' &= a_{22}' = \frac{1}{\alpha_{33}} + \frac{1}{4} \frac{\alpha_{11} + \alpha_{22} - 2\alpha_{12}}{\alpha_{11}\alpha_{22} - \alpha_{12}^2} = G + \frac{1}{4} \frac{E_{11} + E_{22} + 2\nu_{11}E_{22}}{1 - \nu_{11}\nu_{22}} \\
 a_{12}' &= -\frac{1}{\alpha_{33}} + \frac{1}{4} \frac{\alpha_{11} + \alpha_{22} - 2\alpha_{12}}{\alpha_{11}\alpha_{22} - \alpha_{12}^2} = -G + \frac{1}{4} \frac{E_{11} + E_{22} + 2\nu_{11}E_{22}}{1 - \nu_{11}\nu_{22}} \quad (23) \\
 a_{33}' &= \frac{1}{4} \frac{\alpha_{11} + \alpha_{22} + 2\alpha_{12}}{\alpha_{11}\alpha_{22} - \alpha_{12}^2} = \frac{1}{4} \frac{E_{11} + E_{22} - 2\nu_{11}E_{22}}{1 - \nu_{11}\nu_{22}} \\
 * \quad a_{13}' &= a_{23}' = \frac{1}{4} \frac{\alpha_{11} - \alpha_{22}}{\alpha_{11}\alpha_{22} - \alpha_{12}^2} = \frac{1}{4} \frac{E_{22} - E_{11}}{1 - \nu_{11}\nu_{22}}
 \end{aligned}$$

In this particular case

$$a_{11}' = a_{22}' \quad \text{and} \quad a_{13}' = a_{23}' \quad (23a)$$

for both the elastic parameters as for the moduli of elasticity.

It further follows that the constants $a_{13}' = a_{23}'$ disappear, when the principal moduli of elasticity E_{11} and E_{22} are of equal magnitude.

The representation of the buckling loads (determined in the second part of the report) for the particular case $\omega = 45^\circ$ and 135° , respectively, is referred to the two stiffness ratios.

$$\frac{D_{23}'}{D_{22}'} = \frac{a_{23}'}{a_{22}'} \quad \text{and} \quad \frac{D_{33}'}{D_{22}'} = \frac{a_{12}' + 2a_{33}'}{a_{22}'}$$

A limiting value consideration for these two expressions is added at this time.

* This equation was incorrect in the original version of this paper and has been corrected by the NACA Reviewer.

With equations (23) the stiffness ratios $\frac{D_{23}'}{D_{22}'}$ and $\frac{D_{33}'}{D_{22}'}$ can be represented with respect to the moduli of elasticity of the principal axes; they then assume the form

$$\frac{D_{33}'}{D_{22}'} = \frac{-4 \frac{G}{E_{11}} (1 - \nu_{11} \nu_{22}) + \frac{E_{22}}{E_{11}} (3 - 2\nu_{11}) + 3}{4 \frac{G}{E_{11}} (1 - \nu_{11} \nu_{22}) + \frac{E_{22}}{E_{11}} (1 + 2\nu_{11}) + 1} \quad (24)$$

$$\frac{D_{23}'}{D_{22}'} = \frac{\frac{E_{22}}{E_{11}} - 1}{4 \frac{G}{E_{11}} (1 - \nu_{11} \nu_{22}) + \frac{E_{22}}{E_{11}} (1 + 2\nu_{11}) + 1} \quad (24b)$$

The limiting value which $\frac{D_{33}'}{D_{22}'}$ can assume for the limiting case $E_{11} = \infty$ is $\frac{D_{33}'}{D_{22}'} = 3$ and for the limiting case $E_{22} = 0$

$$\frac{D_{33}'}{D_{22}'} = \frac{-4 \frac{G}{E_{11}} (1 - \nu_{11} \nu_{22}) + 3}{4 \frac{G}{E_{11}} (1 - \nu_{11} \nu_{22}) + 1}$$

The modulus of shear can assume values between 0 and ∞ : for $G = 0$ the result is again $\frac{D_{33}'}{D_{22}'} = 3$ and for $G = \infty$, $\frac{D_{33}'}{D_{22}'} = -1$. Likewise, it can be shown that $\frac{D_{33}'}{D_{22}'} = 3$ for $E_{22} = \infty$, and $\frac{D_{33}'}{D_{22}'} = 3$ and -1 , respectively, for $E_{11} = 0$, depending upon whether the modulus G takes the value 0 or ∞ .

The elimination of term $4 \frac{G}{E_{11}} (1 - \nu_{11}\nu_{22})$ from the two equations for $\frac{D_{33}'}{D_{22}'}$ and $\frac{D_{23}'}{D_{22}'}$ leaves a relation between these expressions from which the terms with ν cancel out:

$$4 \frac{D_{23}'}{D_{22}'} = \left(1 + \frac{D_{33}'}{D_{22}'}\right) \frac{\frac{E_{22}}{E_{11}} - 1}{\frac{E_{22}}{E_{11}} + 1}$$

The factor $\left(\frac{E_{22}}{E_{11}} - 1\right) : \left(\frac{E_{22}}{E_{11}} + 1\right)$ assumes values between -1 and 1 , when $\frac{E_{22}}{E_{11}}$ passes through all possible values between 0 and ∞ , so that the limiting values of $\frac{D_{23}'}{D_{22}'}$ follow the equation

$$\frac{D_{23}'}{D_{22}'} = \pm \frac{1 + \frac{D_{33}'}{D_{22}'}}{4} \quad (24c)$$

This investigation indicates that the pair of values $\frac{D_{23}'}{D_{22}'}$ and $\frac{D_{33}'}{D_{22}'}$ exist only in a field that is bounded by the two straight lines

$$\frac{D_{23}'}{D_{22}'} = \frac{1 + \frac{D_{33}'}{D_{22}'}}{4} \quad \text{and} \quad \frac{D_{23}'}{D_{22}'} = -\frac{1 + \frac{D_{33}'}{D_{22}'}}{4}$$

and by the straight line $\frac{D_{33}'}{D_{22}'} = 3$ (fig. 5a).

2. Bending of General-Orthotropic Plates

The theory of bending of general-orthotropic plates follows the classical theory of bending; hence, first of all, the assumptions that the plate thickness is small and that the deflection of the plate itself is small compared to plate thickness.

On the basis of the coordinate system conformable to figure 5b, the intersection moments of the plate read as usual

$$M_x = \int_{-s/2}^{s/2} \sigma_x z \, dz; \quad M_y = \int_{-s/2}^{s/2} \sigma_y z \, dz; \quad M_{xy} = \int_{-s/2}^{s/2} \tau z \, dz \quad (25)$$

The condition of equilibrium between the specific load p of the plate applied at an element and the moments M_x , M_y , and M_{xy} reads:

$$* \quad \frac{\partial^2 M_x}{\partial x^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + p = 0 \quad (26)$$

This equilibrium condition is independent of the elastic properties of the plate, and holds for the isotropic as for the orthotropic plate.

The plate is bent by the moments. Assuming, as usual, a linear strain distribution over the plate thickness, a point of the plate distant by x from the center is, as a result of the curvature of the plate, subjected to the distortion

$$\begin{aligned} \epsilon_x &= -z \frac{\partial^2 w}{\partial x^2} \\ \epsilon_y &= -z \frac{\partial^2 w}{\partial y^2} \\ \gamma &= -2z \frac{\partial^2 w}{\partial x \partial y} \end{aligned} \quad (27)$$

w = deflection of plate.

*Corrected by NACA Reviewer.

Introducing the stresses of the general-orthotropic plate in the expressions (25) for the moments of the plate according to equation (15), while replacing at the same time the distortions appearing in these expressions by the equations (27), gives the moments of the plate, when the integral is extended over the plate width, as:

$$M_x = \int_{-s/2}^{s/2} \sigma_x z \, dz = \int_{-s/2}^{s/2} [a_{11}' \epsilon_x + a_{12}' \epsilon_y + a_{13}' \gamma] z \, dz$$

$$M_x = -\frac{s^3}{12} \left(a_{11}' \frac{\partial^2 w}{\partial x^2} + a_{12}' \frac{\partial^2 w}{\partial y^2} + 2a_{13}' \frac{\partial^2 w}{\partial xy} \right)$$

$$M_y = \int_{-s/2}^{s/2} \sigma_y z \, dz = \int_{-s/2}^{s/2} [a_{21}' \epsilon_x + a_{22}' \epsilon_y + a_{23}' \gamma] z \, dz$$

(28)

$$M_y = -\frac{s^3}{12} \left(a_{12}' \frac{\partial^2 w}{\partial x^2} + a_{22}' \frac{\partial^2 w}{\partial y^2} + 2a_{23}' \frac{\partial^2 w}{\partial xy} \right)$$

$$M_{xy} = \int_{-s/2}^{s/2} \tau z \, dz = \int_{-s/2}^{s/2} [a_{31}' \epsilon_x + a_{32}' \epsilon_y + a_{33}' \gamma] z \, dz$$

$$M_{xy} = -\frac{s^3}{12} \left(a_{13}' \frac{\partial^2 w}{\partial x^2} + a_{23}' \frac{\partial^2 w}{\partial y^2} + 2a_{33}' \frac{\partial^2 w}{\partial xy} \right)$$

The second derivatives of equations (28), formed and entered in the equilibrium condition (26), give the differential equation of the elastic surface of the general-orthotropic plate:

$$\begin{aligned} \frac{s^3}{12} \left[a_{11}' \frac{\partial^4 w}{\partial x^4} + 4a_{13}' \frac{\partial^4 w}{\partial x^3 \partial y} + (2a_{12}' + 4a_{33}') \frac{\partial^4 w}{\partial x^2 \partial y^2} \right. \\ \left. + 4a_{23}' \frac{\partial^4 w}{\partial x \partial y^3} + a_{22}' \frac{\partial^4 w}{\partial y^4} \right] - p = 0 \end{aligned} \quad (29)$$

and, after the abbreviations

$$\begin{aligned}
 D_{11}' &= \frac{s^3}{12} a_{11}' \\
 D_{22}' &= \frac{s^3}{12} a_{22}' \\
 D_{33}' &= \frac{s^3}{12} (a_{12}' + 2a_{33}') \\
 D_{13}' &= \frac{s^3}{12} a_{13}' \\
 D_{23}' &= \frac{s^3}{12} a_{23}'
 \end{aligned} \tag{29a}$$

the final form of the differential equation as

$$D_{11}' \frac{\partial^4 w}{\partial x^4} + 4D_{13}' \frac{\partial^4 w}{\partial x^3 \partial y} + 2D_{33}' \frac{\partial^4 w}{\partial x^2 \partial y^2} + 4D_{23}' \frac{\partial^4 w}{\partial x \partial y^3} + D_{22}' \frac{\partial^4 w}{\partial y^4} = p \tag{30}$$

Compared to the differential equations of the elastic surface of the isotropic and orthotropic plate, respectively, the additive terms with the mixed derivatives

$$\frac{\partial^4 w}{\partial x^3 \partial y} \quad \text{and} \quad \frac{\partial^4 w}{\partial x \partial y^3}$$

appear; they identify the contribution of the elastic parameters a_{13}' and a_{23}' on the absorption of the load p .

For the special-orthotropic plate, D_{13}' and D_{23}' disappear and the remaining constants assume the form indicated by Huber:

$$\begin{aligned} D_{11} &= \frac{s^3}{12} \frac{E_{11}}{1 - \nu_{11}\nu_{22}} \\ D_{22} &= \frac{s^3}{12} \frac{E_{22}}{1 - \nu_{11}\nu_{22}} \\ 2D_{33} &= \frac{s^3}{12} \left(\nu_{11} \frac{E_{22}}{1 - \nu_{11}\nu_{22}} + \nu_{22} \frac{E_{11}}{1 - \nu_{11}\nu_{22}} + 4G \right) \end{aligned} \quad (31)$$

The differential equation of the elastic surface of the special-orthotropic plate reads then

$$D_{11} \frac{\partial^4 w}{\partial x^4} + 2D_{33} \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_{22} \frac{\partial^4 w}{\partial y^4} = p \quad (32)$$

For the isotropic plate, when noting that

$$E_{11} = E_{22} = E$$

$$\nu_{11} = \nu_{22} = \nu$$

$$G = \frac{E(1 - \nu)}{2(1 - \nu^2)}$$

can be put in (31), it finally gives

$$D_{11} = D_{22} = D_{33} = D = \frac{s^3}{12} \frac{E}{1 - \nu^2}$$

and the differential equation can be written in the conventional form

$$\frac{\partial^4 w}{\partial x^4} + \frac{2\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{p}{D} \quad (33)$$

3. The Internal Flexural Strain Energy of the General-Orthotropic Plate

The internal strain energy of a plate can be generally represented by the relation

$$2A_1 = \iiint [\sigma_x \epsilon_x + \sigma_y \epsilon_y + \tau \gamma] dx dy dz \quad (34)$$

If the stresses σ_x , σ_y , τ in the bracketed term of equation (1) are replaced according to equation (13) by the corresponding distortions, the equation for the general-orthotropic plate can also be written in the form

$$\begin{aligned} 2A_1 = \iiint & [a_{11} \epsilon_x^2 + a_{12} \epsilon_y \epsilon_x + a_{13} \gamma \epsilon_x \\ & + a_{21} \epsilon_x \epsilon_y + a_{22} \epsilon_y^2 + a_{23} \gamma \epsilon_y \\ & + a_{31} \gamma \epsilon_x + a_{32} \epsilon_y \gamma + a_{33} \gamma^2] dx dy dz \end{aligned} \quad (35)$$

If the plate is stressed in bending by moments, the flexural distortions ϵ_x , ϵ_y , and γ can be expressed again by the corresponding plate curvatures, (equation 21), and gives, after integration with respect to plate thickness s , the internal flexural strain energy of the general orthotropic plate

$$\begin{aligned} A_1 = \frac{s^3}{24} \iint & \left[a_{11} \left(\frac{\partial^2 w}{\partial x^2} \right)^2 + 2a_{12} \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} + a_{22} \left(\frac{\partial^2 w}{\partial y^2} \right)^2 + 4a_{13} \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial xy} \right. \\ & \left. + 4a_{23} \frac{\partial^2 w}{\partial y^2} \frac{\partial^2 w}{\partial xy} + 4a_{33} \left(\frac{\partial^2 w}{\partial xy} \right)^2 \right] dx dy \end{aligned} \quad (36)$$

For isotropic material, $a_{11}' = a_{22}' = a_{11}$; in addition, the constants a_{13}' and a_{23}' disappear and the flexural energy follows as:

$$A_1 = \frac{s^3}{24} \iint \left\{ a_{11} \left[\left(\frac{\partial^2 w}{\partial x^2} \right)^2 + \left(\frac{\partial^2 w}{\partial y^2} \right)^2 \right] + 2a_{12} \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} + 4a_{33} \left(\frac{\partial^2 w}{\partial xy} \right)^2 \right\} dx dy \quad (37)$$

If, instead of the constants a_{11}' , a_{12} , and a_{33} the customary reciprocal designations are used again, equation (37) can be replaced by the known expression for the internal flexural strain energy of the isotropic plate

$$A_1 = \frac{s^3}{12} G \iint \left\{ \frac{1}{1 - \gamma} \left[\left(\frac{\partial^2 w}{\partial x^2} \right)^2 + \left(\frac{\partial^2 w}{\partial y^2} \right)^2 \right] + \frac{2}{1 - \gamma} \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} + 2 \left(\frac{\partial^2 w}{\partial xy} \right)^2 \right\} dx dy \quad (37a)$$

4. Experimental Determination of the Moduli of Elasticity

of the General-Orthotropic Plate

The elastic properties of the general-orthotropic plate are characterized by six moduli of elasticity. Inasmuch as, according to equation (6), the six moduli of the general-orthotropic plate can be determined from the moduli α_{11} , α_{22} , α_{12} , and α_{33} for every direction angle ω , the determination of these four moduli is by itself sufficient for a complete identification of the elastic properties of the plate. But for an experimental check on the relationship between the moduli of elasticity represented in equation (6) and the direction angle ω it seems appropriate to have a method available which permits the determination of all six moduli of the plate.

(a) Pure Axial Stress Condition - Pure Shearing-Stress Condition

Nadai (reference 8, pp. 355 ff) pointed out that two stress attitudes, the case of pure axial bending and the case of pure twisting of a plate (figs. 6, 7), are particularly suitable for determining the two moduli of elasticity of isotropic plate-shaped bodies. The boundary conditions in both cases are easily established and the deflections of the plate readily indicated, since the central area of the plate assumes a geometrically simple shape in these loading conditions.

The determination of the moduli of elasticity from bending and torsion tests, especially for plywood plates, which by reason of their inhomogeneous structure over the plate thickness manifest different elasticity moduli in bending than in plane stress, (see section 5) has the advantage to the extent that the decisive moduli, necessary for the investigation of the plate stability, are already obtained this way for bending and torsion stress.

Nadai's method for isotropic plates can also be extended to include the general-orthotropic plate and the six moduli of this plate computed with it.

A relation between the stresses introduced by the loading and the deflection of the plate is obtained by equating the distortions ϵ_x , ϵ_y , and γ according to equations (1) and (27) to each other:

$$\begin{aligned}\epsilon_x &= -z \frac{\partial^2 w}{\partial x^2} = \alpha_{11}' \sigma_x + \alpha_{12}' \sigma_y + \alpha_{13}' \tau \\ \epsilon_y &= -z \frac{\partial^2 w}{\partial y^2} = \alpha_{12}' \sigma_x + \alpha_{22}' \sigma_y + \alpha_{23}' \tau \\ \gamma &= -2z \frac{\partial^2 w}{\partial xy} = \alpha_{13}' \sigma_x + \alpha_{23}' \sigma_y + \alpha_{33}' \tau\end{aligned}\tag{38} - (1)$$

On assuming a uniform tension stress and shearing stress attitude, the stresses in the plate are constant.

Putting for the stresses in the outer layer $z = \frac{s}{2}$ of the plate $\sigma_x = \sigma_{x0}$; $\sigma_y = \sigma_{y0}$ and $\tau = \tau_0$, equation (38) can be written as

$$\begin{aligned}\frac{\partial^2 w}{\partial x^2} &= -\frac{2}{s} (\alpha_{11}' \sigma_{x0} + \alpha_{12}' \sigma_{y0} + \alpha_{13}' \tau_0) \\ \frac{\partial^2 w}{\partial y^2} &= -\frac{2}{s} (\alpha_{12}' \sigma_{x0} + \alpha_{22}' \sigma_{y0} + \alpha_{23}' \tau_0) \\ \frac{\partial^2 w}{\partial xy} &= -\frac{1}{s} (\alpha_{13}' \sigma_{x0} + \alpha_{23}' \sigma_{y0} + \alpha_{33}' \tau_0)\end{aligned}\tag{38a}$$

The general form of a function which satisfies equations (38a), is

$$w = -\frac{\sigma_{x0}}{s} (\alpha_{11}'x^2 + \alpha_{12}'y^2 + \alpha_{13}'xy) - \frac{\sigma_{y0}}{s} (\alpha_{12}'x^2 + \alpha_{22}'y^2 + \alpha_{23}'xy) \\ - \frac{\tau_0}{s} (\alpha_{13}'x^2 + \alpha_{23}'y^2 + \alpha_{33}'xy) + c_1x + c_2y + c_3 \quad (39)$$

where the remaining terms $c_1x + c_2y + c_3$ are disregarded as unessential since they merely define the position of the plate in space. Limited to the case that the plate is deformed in the plane only by a moment M_x , it results in a pure longitudinal stress condition and, since σ_{y0} and τ_0 disappear, the equation of the elastic surface - after putting

$$\sigma_{x0} = \frac{M_x}{W} = \frac{6M_x}{bs^2} = \frac{6m_x}{s^2}$$

reads

$$w = \frac{6m_x}{s^3} (\alpha_{11}'x^2 + \alpha_{12}'y^2 + \alpha_{13}'xy) \quad (40a)$$

If the plate is stressed by a moment M_y only,

$$w = \frac{6m_y}{s^3} (\alpha_{12}'x^2 + \alpha_{22}'y^2 + \alpha_{23}'xy) \quad (40b)$$

If the plate is stressed by M_{xy} moments only, $\sigma_{x0} = \sigma_{y0} = 0$, and deflection function w is

$$w = \frac{6m_{xy}}{s^3} (\alpha_{13}'x^2 + \alpha_{23}'y^2 + \alpha_{33}'xy) \quad (40c)$$

On the isotropic plate, a pure longitudinal stress condition, that is, a load due to bending moments M_x or M_y , produces pure bending, a load in pure shear, that is, a load due to twisting moments M_{xy} , produces a pure twist. (The term with xy in equation (40a) and (40b) disappears for the isotropic plate, because $\alpha_{13}' = \alpha_{23}' = 0$; the elastic surface is therefore symmetrical to the coordinate axes, that is, pure bending results. The terms with x^2 and y^2 in equation (40c) disappear, the elastic surface is antisymmetrical to the coordinate axes, hence pure twist). But on the general-orthotropic plate, bending moments, as well as twisting moments, even if applied separately, induce bending and twist simultaneously.

Equations 40a to 40c indicate further that the pure longitudinal stress attitude is suitable for determining the moduli

$$\alpha_{11}', \alpha_{22}', \alpha_{12}', \alpha_{13}', \alpha_{23}'$$

while the pure shearing stress permits the moduli $\alpha_{13}', \alpha_{23}', \alpha_{33}'$ to be measured. Consequently, moduli α_{13}' and α_{23}' can be obtained by either one of the two conditions, but the modulus of rigidity α_{33}' only by the shearing stress, the moduli of elasticity and transverse contraction moduli $\alpha_{11}', \alpha_{22}'$ and α_{12}' only by means of the pure longitudinal stress condition. Experimentally, the task can be confined to measuring the moduli $\alpha_{11}', \alpha_{22}'$ and α_{12}' for the longitudinal stress, and the moduli $\alpha_{13}', \alpha_{23}'$ and α_{33}' for the shearing stress condition.

(b) Experimental Determination of the Moduli of Elasticity

The measurements for the determination of moduli $\alpha_{11}', \alpha_{22}'$, and α_{12}' are, according to Nadai's proposal, made on a plate strip supported at three points. The bending moment producing the pure longitudinal stress in the plate is applied by means of linkages which load the plate in three points (fig. 8). The deflection is obtained from the difference of the readings of three dial gages mounted on the x axis. By equation (40a)

$$w_x = \frac{6m_x}{s^3} \alpha_{11}' x^2$$

for the x axis ($y = 0$), hence for modulus α_{11}' , when the two outer dial gages are spaced at distance x from the central gage,

$$\alpha_{11}' = \frac{w_x s^3}{6m_x} \frac{1}{x^2} \quad (41)$$

A corresponding measurement on the y axis gives

$$\alpha_{12}' = \frac{w_y s^3}{6m_x} \frac{1}{y^2} \quad (42)$$

while appropriate variation of the direction of the principal axes of the plate with respect to the coordinate axes gives

$$\alpha_{22}' = \frac{w_y s^3}{6m_y} \frac{1}{y^2} \quad (43)$$

and

$$\alpha_{12}' = \frac{w_x s^3}{6m_y} \frac{1}{x^2} \quad (42)$$

respectively.

The measurements for the determination of the moduli α_{13}' , α_{23}' , and α_{33}' are made on a square plate which is loaded according to figure 9. The loading produces a pure shearing stress since, (reference 6, pp. 299 ff) the twisting moments m_{xy} occurring at the boundary of the plate can be replaced by two equal and opposite transverse forces Q normal to the plane of the plate, in the corners of the plate. The total single force in a corner is

$$P = 2Q = 2m_{xy}$$

The measurement of moduli α_{13}' and α_{23}' is similar to that of α_{11}' and α_{12}' .

If the deflection of the plate is measured along the x axis
($y = 0$)

$$w_x = \frac{6m_{xy}}{s^3} \alpha_{13}' x^2 = \frac{3P}{s^3} \alpha_{13}' x^2 \quad (44)$$

$$\alpha_{13}' = \frac{w_x s^3}{3Px^2}$$

follows from equation (40c), where x again denotes the distance of the two outer gages from the central dial gage.

Measuring the deflection w_y along the y axis gives the modulus

$$\alpha_{23}' = \frac{w_y s^3}{3Py^2} \quad (45)$$

The modulus of rigidity is defined by the measurement of the deflection of the plate along its diagonals (fig. 9). According to equation (40c); the deflection of the plate in the zero point on the diagonal 1 gives - the outer dial gages being mounted at the points $x = y = b$, and $x = y = -b$

$$w_1 = \frac{3P}{s^3} (\alpha_{13}' b^2 + \alpha_{23}' b^2 + \alpha_{33}' b^2)$$

and on diagonal 2 with the corresponding points $x = b$, $y = -b$, and $x = -b$, $y = b$:

$$w_2 = \frac{3P}{s^3} (\alpha_{13}' b^2 + \alpha_{23}' b^2 - \alpha_{33}' b^2)$$

Forming the difference leaves

$$w_1 - w_2 = \frac{6Pb^2}{s^3} \alpha_{33}'$$

and hence the modulus of rigidity α_{33}' as

$$\alpha_{33}' = \frac{(w_1 - w_2)s^3}{6Pb^2} \quad (46)$$

5. Plywood as Orthotropic Plate

(a) Application of Theory of Elasticity of the Homogeneous General-Orthotropic Plate to Plywood

Since the results of the present report in their practical application are primarily intended for plywood, it is necessary to explore the extent to which the theory of elasticity of the homogeneous general-orthotropic plate is applicable to plywood.

Whereas the ordinary wood veneer can, with some justification, be regarded as a plate of orthogonal, anisotropic homogeneous material to which the theory of elasticity, and particularly also the theory of bending of the orthotropic plate, can be applied, the combination of single veneers at 90° to each other into plywood also presents an orthotropic plate, but inhomogeneous over the plate thickness. In consequence, if the relations applicable to homogeneous material are formally retained, the moduli of elasticity of a plywood plate assume, in part, different values for stresses in its plane than for stresses which tend to deflect the plate out of its plane.

For a special-orthotropic plate consisting of three identically thick beech veneer plies, for example, whose veneers by themselves show the moduli of elasticity $E_1 = 180,000 \text{ kg/cm}^2$ and $E_2 = 5,000 \text{ kg/cm}^2$ respectively, the assumption of equal strain of the glued laminations under longitudinal stresses in the plane of the plate, with transverse

contraction disregarded, gives the modulus of elasticity E_x in x direction (fig. 10) as

$$E_x = \frac{\sum EF}{F} = \frac{\frac{2s}{3} E_1 + \frac{s}{3} E_2}{s} = \frac{2E_1 + E_2}{3} = 121,700 \text{ kg/cm}^2 *$$

and in y direction as

$$E_y = \frac{E_1 + 2E_2}{3} = 63,300 \text{ kg/cm}^2 *$$

But, stressed in bending by moments about the x or y axis, the retention of linear strain distribution over the plate thickness gives the moduli of elasticity

$$E_x = \frac{\sum EJ}{J} = \frac{E_1 \left[2\left(\frac{s}{3}\right)^3 + \frac{2}{12}\left(\frac{s}{3}\right)^3 \right] + \frac{E_2}{12}\left(\frac{s}{3}\right)^3}{\frac{s^3}{12}}$$

$$E_x = \frac{26}{27} E_1 + \frac{1}{27} E_2 = 173,500 \text{ kg/cm}^2$$

and for E_y

$$E_y = \frac{1}{27} E_1 + \frac{26}{27} E_2 = 11,500 \text{ kg/cm}^2$$

that is, substantially different values than for the plane stress. Similarly, the transverse contraction factors in plane stresses are different from those for bending stresses of the plate, as will be shown later.

The modulus of rigidity of the special-orthotropic plate, on the other hand, has, for pure shear in the plane, the same magnitude as for

*

These values were altered by NACA Reviewer.

the twisting stress of the plate due to twisting moments about the x or y axis, since the modulus of rigidity of each lamination is of the same magnitude, regardless of whether the grain of the veneer is in x direction or at right angles to it (fig. 11). Therefore, the special-orthotropic plate is, as far as the modulus of rigidity is concerned, homogeneous over the plate thickness. The modulus of rigidity of the plywood plate should, in consequence, agree with that of the veneer, but on account of the bonding of the laminations the modulus of rigidity of the plywood plate compared to the individual veneer, increases according to Hertel (reference 7), from about $G = 7000\text{kg/cm}^2$ to $G = 10000$ to 12000kg/cm^2 , depending upon the nature and quality of bonding.

After the four moduli of elasticity for plane or bending stress of the special-orthotropic plate have been found - whether by calculation from the moduli of elasticity of the veneer (cf. next chapter) or by direct strain and bending tests on the plywood plate - the moduli of elasticity of the general-orthotropic plate for any angle ω can be obtained by the equations (6), so that the theory of the general-orthotropic plate remains unrestrictedly applicable also to the plywood plate. The extent to which the assumption of linear strain distribution is in agreement with actual conditions must be verified by comparison of the calculation with the experimental results.

(b) Effect of Plywood Construction on the Moduli of Elasticity of the Special-Orthotropic Plywood Plate

Investigations concerning the effect of plywood construction on the moduli of elasticity of the special-orthotropic plate have been made by Hertel (reference 7). Beyond the results of this work, it is shown in the following that for the moduli α_{11} , α_{22} , and α_{12} simple approximation formulas can be set up for their dependence on the plate construction, and that the formulas derived for the plane stress of the plywood plate can also be applied to the plywood plate under bending stress.

(α) Plane stress of plywood plate. - On decomposing the plywood plate into the portion on longitudinally and transversely directed veneers (fig. 12), the strain equations for each veneer portion - the moduli, stresses, and distortions of the longitudinally directed plies being indicated by superscript (1), those of the transversely directed by superscript (2) - can be written as follows:

$$\begin{aligned}\epsilon_x^{(1)} &= \alpha_{11}^{(1)} \sigma_x^{(1)} + \alpha_{12}^{(1)} \sigma_y^{(1)} \\ \epsilon_y^{(1)} &= \alpha_{12}^{(1)} \sigma_x^{(1)} + \alpha_{22}^{(1)} \sigma_y^{(1)}\end{aligned}\tag{47}$$

and

$$\begin{aligned}\epsilon_x^{(2)} &= \alpha_{11}^{(2)}\sigma_x^{(2)} + \alpha_{12}^{(2)}\sigma_y^{(2)} \\ \epsilon_y^{(2)} &= \alpha_{12}^{(2)}\sigma_x^{(2)} + \alpha_{22}^{(2)}\sigma_y^{(2)}\end{aligned}\tag{48}$$

and for the plywood plate

$$\begin{aligned}\epsilon_x &= \alpha_{11}\sigma_x + \alpha_{12}\sigma_y \\ \epsilon_y &= \alpha_{12}\sigma_x + \alpha_{22}\sigma_y\end{aligned}\tag{49}$$

The shearing strain is disregarded, since, according to the foregoing, the modulus of rigidity G of the special-orthotropic plate is independent of the construction of the plywood.

From equilibrium conditions at the plywood plate, with consideration of the two veneer portions, follow

$$\sigma_x^{(1)}F_1 + \sigma_x^{(2)}F_2 = \sigma_x F\tag{50}$$

$$\sigma_y^{(1)}F_1 + \sigma_y^{(2)}F_2 = \sigma_y F$$

or, when putting $\frac{F_2}{F_1} = D$

$$\sigma_x^{(1)} + \sigma_x^{(2)}D = \sigma_x(1 + D)\tag{50a}$$

$$\sigma_y^{(1)} + \sigma_y^{(2)}D = \sigma_y(1 + D)$$

On assuming uniform strain over the cross section of the plywood, the condition

$$\begin{aligned}\epsilon_x^{(1)} &= \epsilon_x^{(2)} = \epsilon_x \\ \epsilon_y^{(1)} &= \epsilon_y^{(2)} = \epsilon_y\end{aligned}\tag{51}$$

is applicable.

Observing that

$$\alpha_{11}^{(1)} = \alpha_{22}^{(2)}$$

$$\alpha_{11}^{(2)} = \alpha_{22}^{(1)}$$

and

$$\alpha_{12}^{(1)} = \alpha_{12}^{(2)}$$

equations (51) in conjunction with (50a), (47), and (48), the stresses

$$\sigma_x^{(2)} \quad \text{and} \quad \sigma_y^{(2)}$$

can be represented with respect to the moduli of elasticity of the veneer portion (1) and to the stresses σ_x and σ_y . The moduli of elasticity α_{11} and α_{12} follow then from the relation

$$\epsilon_x = \epsilon_x^{(2)} = \alpha_{11}\sigma_x + \alpha_{12}\sigma_y = \alpha_{22}^{(1)}\sigma_x^{(2)} + \alpha_{12}^{(1)}\sigma_y^{(2)}$$

The moduli α_{22} and likewise α_{12} can be defined in similar manner from the equation

$$\epsilon_y = \epsilon_y^{(2)} = \alpha_{12}\sigma_x + \alpha_{22}\sigma_y = \alpha_{12}^{(1)}\sigma_x^{(2)} + \alpha_{11}^{(1)}\sigma_y^{(2)}$$

The calculation gives for α_{11}

$$\alpha_{11} = \alpha_{11}^{(1)}(1 + D) \frac{\frac{\alpha_{11}^{(1)}}{\alpha_{22}^{(1)}} + D - \frac{\alpha_{12}^{(1)2}}{\alpha_{11}^{(1)}\alpha_{22}^{(1)}} \left(1 - \frac{\alpha_{11}^{(1)}}{\alpha_{22}^{(1)}}\right) - \frac{\alpha_{12}^{(1)2}}{\alpha_{12}^{(1)2}}(1 + D)}{\left(1 + \frac{\alpha_{11}^{(1)}}{\alpha_{22}^{(1)}} D\right) \left(\frac{\alpha_{11}^{(1)}}{\alpha_{22}^{(1)}} + D\right) - \left(\frac{\alpha_{12}^{(1)}}{\alpha_{22}^{(1)}}\right)^2 (1 + D)^2} \quad (52)$$

and correspondingly for α_{12} :

$$\alpha_{12} = \alpha_{12}^{(1)}(1 + D) \frac{(1 + D) \left(\frac{\alpha_{11}^{(1)}}{\alpha_{22}^{(1)}} - \frac{\alpha_{12}^{(1)2}}{\alpha_{22}^{(1)2}} \right)}{\left(1 + \frac{\alpha_{11}^{(1)}}{\alpha_{22}^{(1)}} D\right) \left(\frac{\alpha_{11}^{(1)}}{\alpha_{22}^{(1)}} + D\right) - \left(\frac{\alpha_{12}^{(1)}}{\alpha_{22}^{(1)}}\right)^2 (1 + D)^2} \quad (53)$$

According to Hertel's measurements on beech veneer, the moduli of elasticity α have the following values:

$$\alpha_{11}^{(1)} = 5.56 \times 10^{-6} \frac{\text{cm}^2}{\text{kg}}$$

$$\alpha_{22}^{(1)} = 200 \times 10^{-6} \frac{\text{cm}^2}{\text{kg}}$$

$$\alpha_{12}^{(1)} = -2.5 \times 10^{-6} \frac{\text{cm}^2}{\text{kg}}$$

Since the transverse contraction modulus $\alpha_{12}^{(1)}$ is small compared to modulus $\alpha_{22}^{(1)}$, the terms with $\left(\frac{\alpha_{12}^{(1)}}{\alpha_{22}^{(1)}}\right)^2$ in equations (52) and (53) can be crossed out, thus leaving the simplified equations

$$\alpha_{11} = \alpha_{11}^{(1)}(1 + D) \frac{\frac{\alpha_{11}^{(1)}}{\alpha_{22}^{(1)}} + D - \frac{\alpha_{12}^{(1)^2}}{\alpha_{11}^{(1)}\alpha_{22}^{(1)}} D \left(1 - \frac{\alpha_{11}^{(1)}}{\alpha_{22}^{(1)}}\right)}{\left(1 + \frac{\alpha_{11}^{(1)}}{\alpha_{22}^{(1)}} D\right) \left(\frac{\alpha_{11}^{(1)}}{\alpha_{22}^{(1)}} + D\right)} \quad (52a)$$

and

$$\alpha_{12} = \alpha_{12}^{(1)}(1 + D) \frac{\frac{\alpha_{11}^{(1)}}{\alpha_{22}^{(1)}} (1 + D)}{\left(1 + \frac{\alpha_{11}^{(1)}}{\alpha_{22}^{(1)}} D\right) \left(\frac{\alpha_{11}^{(1)}}{\alpha_{22}^{(1)}} + D\right)} \quad (53a)$$

Elimination of the term with $\frac{\alpha_{12}(1)^2}{\alpha_{11}(1)\alpha_{22}(1)}$ in equation (52), leaves

$$\alpha_{11} = \alpha_{11}(1) \frac{(1 + D)}{1 + \frac{\alpha_{11}(1)}{\alpha_{22}(1)} D} \quad (52b)$$

The effect which the modulus of transverse contraction exerts on the modulus of elasticity of the plywood is quite small and amounts to less than 0.5 percent, according to comparisons of the approximation formula (52b) with the exact formula (52). The modulus of transverse contraction of the plywood is defined by (53a) with approximately the same degree of accuracy; with the use of (53a) and (52b), this modulus can also be written as

$$\nu = -\frac{\alpha_{12}}{\alpha_{11}} = \nu_{11}(1) \frac{1 + D}{1 + \frac{\alpha_{22}(1)}{\alpha_{11}(1)} D} \quad (54)$$

The corresponding approximate equations for the moduli α_{22} and $\nu_{22} = -\frac{\alpha_{12}}{\alpha_{22}}$ are

$$\alpha_{22} = \alpha_{22}(1) \frac{1 + D}{1 + \frac{\alpha_{22}(1)}{\alpha_{11}(1)} D} \quad (55)$$

and

$$\nu_{22} = \nu_{22}(1) \frac{1 + D}{1 + \frac{\alpha_{11}(1)}{\alpha_{22}(1)} D} \quad (56)$$

(β) Bending stress of the plywood plate.— The fact that the bent plywood plate has different moduli from the plane stressed plate can be taken into account in simple manner.

Assuming linear strain distribution over the plate thickness, the equations (47) to (49) apply also to the bending strain of the laminations, if the elongation of the laminations is referred to that with or across the grain, (fig. 13) (say, to the top boundary layer, for example). The elongations of veneer portions in this boundary layer must be equal to each other (equation 51). The equilibrium conditions for bending follow from the consideration that the sum of the portions of the bending moment which the veneers support must be equal to the total bending moment applied at the plywood plate

$$\begin{aligned}\sigma_x^{(1)}W_1 + \sigma_x^{(2)}W_2 &= \sigma_x W \\ \sigma_y^{(1)}W_1 + \sigma_y^{(2)}W_2 &= \sigma_y W\end{aligned}\tag{57}$$

where W_1 represents the portions with the grain and W_2 those across the grain to the section modulus per unit length (fig. 13)

$$W_1 = \frac{\int_{-s/2}^{s/2} \eta^2 dF \text{ (with the grain)}}{s/2}$$

$$W_2 = \frac{\int_{-s/2}^{s/2} \eta^2 dF \text{ (across the grain)}}{s/2}$$

and W is corresponding to the homogeneous material

$$W = \frac{s^2}{6}$$

If, as for the plane stress, the factor $D = \frac{W_2}{W_1}$ is introduced, equation (57) takes the same form as equation (50a) and all the relations derived for the plane stress apply to the bent plate, too; however, it should be borne in mind that in bending for one and the same plywood plate, a different factor D is used for determining the moduli of elasticity from that for plane stress. To illustrate: For a plywood plate built up of three identical veneers in plane stress, $D_e = 0.5$, as against $D_b = 0.0385$ in bending; for a plywood plate of five identical plies, $D_e = 0.667$, as against $D_b = 0.263$ (Table 1).

Figure 14 represents the moduli of elasticity E_{11} , E_{22} , and the transverse contraction factors ν_{11} , ν_{22} plotted against the factor D which characterizes the construction of the plywood. By this D any desired construction of plywood, hence also plywood consisting of plies of different thicknesses, can be defined. The D factors for plywood of various construction, but identical ply thicknesses (3, 5, 7, 11, ∞ plies) under plane and flexural stress are shown on the abscissa. The calculation for determining the moduli of the plywood was based on Hertel's test data for beech veneers. According to it, the following averages hold for the veneer:

$$E_{11} = 180,000 \text{ kg/cm}^2$$

$$E_{22} = 5,000 \text{ kg/cm}^2$$

$$\nu_{11} = 0.45$$

$$\nu_{22} = 0.0125$$

$$G = 7000 \text{ kg/cm}^2$$

The results of the calculation are included in Table 1. The comparison of the moduli obtained by approximation with those obtained by the exact equations discloses only minor differences. Table 1 and figure 14 also contain the moduli of elasticity for $\omega = 45^\circ$, the determination of which is given elsewhere.

(c) Effect of Angle ω on the Moduli of Elasticity

Wood veneer and — according to the arguments of the preceding section — plywood as well may be regarded as examples for anisotropic (orthotropic) material. Knowing the moduli of elasticity along the principal axes, the application of equation (6) permits the representation of the change in the moduli with the variation of angle ω .

(a) Veneer.— Regarding the moduli of elasticity in the principal axes directions of beech veneers, Hertel's test data are available. Figure 15 shows the moduli of elasticity α for the veneer with these principal axes values plotted against the angle ω .

It is seen that modulus α_{11}' , α_{22}' , α_{12}' , and α_{33}' have a period of 90° , while modulus α_{13}' and α_{20}' repeat only after 180° . The values of α_{13}' and α_{23}' disappear for 0° and 90° (that is, for special-orthotropic plates) and agree for $\omega = 45^\circ$ and 135° , respectively.

The marked anisotropy of beech veneer is particularly apparent on moduli α_{11}' and α_{22}' , which manifest very great variations with the change of ω .

The moduli α_{11}' , α_{12}' , and α_{33}' are shown again in figure 16, but in the reciprocal manner of writing $E_{11}' = \frac{1}{\alpha_{11}'}$ and $\nu_{11}' = -\frac{\alpha_{12}'}{\alpha_{11}'}$.

The modulus of rigidity decreases within narrow limits and assumes the

minimum value of $G = 4750 \frac{\text{kg}}{\text{cm}^2}$ at $\omega = 45^\circ$. Of interest is the modulus

of transverse contraction ν_{11}' , which is positive in the range of the principal axes directions, and negative in the range of $\omega = 11.5^\circ$ to 78.5° , that is, a veneer strip whose angle ω lies in this range becomes wider under tensile stress, a phenomenon experimentally observed by Hertel at $\omega = 45^\circ$. Besides the measurements of modulus E_{11} and ν_{11} at $\omega = 0^\circ$, Hertel also made some measurements of the modulus

at $\omega = 45^\circ$, which showed $E_{11}' = 11,000 \frac{\text{kg}}{\text{cm}^2}$ and $\nu' = -0.185$. These

values indicate good agreement with the computed values of $E_{11}' = 11,700 \frac{\text{kg}}{\text{cm}^2}$ and $\nu' = -0.17$.

(β) Plywood.— With Hertel's moduli of elasticity for the single veneer as basis, the calculation gives, according to the arguments of Section II, 5b, the moduli of the principal axes of the plywood reproduced in table 1, a distinction being necessary between the moduli for bending and plane stress. The modulus of rigidity of the plywood is increased to $10,500 \frac{\text{kg}}{\text{cm}^2}$ as a result of the bonding of the laminations, as against $7,000 \frac{\text{kg}}{\text{cm}^2}$ for the nonglued veneer.

Figures 17 and 18 represent the modulus α' for plywood of various construction plotted against angle ω , and figures 19 and 20 the corresponding reciprocal values for plywood plates stressed in plane and in bending.

The equalization of modulus of elasticity E_{11} and E_{22} at $\omega = 0$ and 90° as a result of the cross laminations compared to the simple veneer is readily apparent; the moduli approach each other with increasing number of plies. A balance of the moduli of elasticity in the range of 45° angle with the E moduli for 0° and 90° is, however, unobtainable on laminations placed at right angles to each other (orthotropic plywood), the E modulus drops at $\omega = 45^\circ$ to about 25 to 30 percent of the values E_{11} at $\omega = 0^\circ$, depending upon the number of plies. An equalization of this range could be obtained also by a plywood which contained diagonal laminations in addition to those at right angles to each other. But such plywood has found no practical application up to now.

The modulus of rigidity G for orthotropic plywood shows the typical rise from $10,500 \frac{\text{kg}}{\text{cm}^2}$ at $\omega = 0^\circ$ and 90° to about $40,000 \frac{\text{kg}}{\text{cm}^2}$ at $\omega = 45^\circ$.

The great gain in shearing strength afforded from the use of plywood plates having their grain diagonal for shear trusses over normally built-up plates is readily apparent.

The diagrams indicate further that the modulus E_{11}' for plane stress varies within comparatively narrow limits with the change in number of plies; in the range of $\omega = 45^\circ$ particularly, the variation of the E_{11}' modulus is scarcely noticeable.

The customary method in practice, independent of the construction of the plywood, to figure with a modulus of elasticity of $100,000 \frac{\text{kg}}{\text{cm}^2}$ in

longitudinal direction ($\omega = 0$), of $300,000 \frac{\text{kg}}{\text{cm}^2}$ ($\omega = 45^\circ$) in diagonal

direction and of $70,000 \frac{\text{kg}}{\text{cm}^2}$ in transverse direction ($\omega = 90^\circ$) appears to be justified to some extent according to the values of figure 19.

On the other hand, the moduli of elasticity E_{11} in bending for $\omega = 0$ and 90° are scattered over a wide range; especially the values for three-ply plywood differ considerably from the others. It therefore seems no longer permissible to introduce fixed values for the moduli of elasticity independent of the construction of the plywood, when plywood plates stressed in bending are involved.

Even the values of G' in plane stressed plywood are not essentially dependent on the number of plies, so that here also the value of $40,000 \frac{\text{kg}}{\text{cm}^2}$ used in practice independent of the construction seems justifiable.

But the modulus of rigidity in flexurally stressed plywood plates is, for $\omega = 45^\circ$, very much dependent on the construction of the plate. An interesting particular case is represented by the three-ply plywood which, independent of the angle ω , has a constant modulus of rigidity of $G' = 10,500 \frac{\text{kg}}{\text{cm}^2}$.

The dependence of the moduli G' , $E_{11}' = E_{22}'$, $\nu_{11}' = \nu_{22}'$ for diagonally oriented plywood plates ($\omega = 45^\circ$) on the construction factor D is also apparent in figure 14, which shows the minor change of the moduli, especially of modulus E' in plane stressed plates.

III. THEORY OF BUCKLING OF THE GENERAL-ORTHOTROPIC PLATE

A. EXACT SOLUTION OF THE PROBLEM

1. The Stability Condition

If an infinitely long, general-orthotropic, homogeneous and elastic plate strip is stressed at its edges by uniformly distributed axial forces N_x and N_y , or shearing forces N_{xy}' , respectively, (fig. 21), these forces produce components at buckling w of the plate, which are at right angles to the plate and of magnitude (reference 6, p. 305):

$$-p = N_x \frac{\partial^2 w}{\partial x^2} + 2N_{xy}' \frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial y^2} \quad (59)$$

If this loading of the plate by the external forces (through corresponding increase of the forces) reaches exactly the value p , which corresponds to the value p as a result of the internal resistance against bending according to equation (30), the plate becomes unstable, that is, it collapses into corrugations or wrinkles.

For this extreme case of stability, the differential equation of the elastic surface becomes, according to equation (30):

$$D_{11}' \frac{\partial^4 w}{\partial x^4} + 4D_{13}' \frac{\partial^4 w}{\partial x^3 \partial y} + 2D_{33}' \frac{\partial^4 w}{\partial x^2 \partial y^2} + 4D_{23}' \frac{\partial^4 w}{\partial x \partial y^3} + D_{22}' \frac{\partial^4 w}{\partial y^4} + N_x \frac{\partial^2 w}{\partial x^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial y^2} = 0 \quad (60)$$

The characteristic values of this differential equation give the desired critical loads at which the plate begins to become unstable, the characteristic functions w corresponding to these characteristic values, the buckling forms of the plate.

2. Formula for Solving the Differential Equation

For the infinitely long strip, the solution w of the differential equation in x direction must be a pure periodic function, in which case it is then possible also to apply to the extended differential equation of the general-orthotropic plate the Southwell-Skan formula (reference 4) for the isotropic plate:

$$w = e^{i\kappa \frac{x}{a}} e^{i\lambda \frac{y}{a}} \quad (70)$$

where κ is a real quantity.

The formula (70), introduced in the differential equation, gives the characteristic equation

$$\lambda^4 + \frac{4D_{23}'}{D_{22}'} \kappa \lambda^3 + \left(\frac{2D_{33}'}{D_{22}'} \kappa^2 - \frac{N_y a^2}{D_{22}'} \right) \lambda^2 + \left(\frac{4D_{13}'}{D_{22}'} \kappa^3 - \frac{2N_{xy} a^2}{D_{22}'} \kappa \right) \lambda + \frac{D_{11}'}{D_{22}'} \kappa^4 - \frac{N_x a^2}{D_{22}'} \kappa^2 = 0 \quad (71)$$

The characteristic equation gives four roots λ_n , so that for given κ the complete integral of the differential equation reads:

$$w = e^{i\kappa \frac{x}{a}} \left[C_1 e^{i\lambda_1 \frac{y}{a}} + C_2 e^{i\lambda_2 \frac{y}{a}} + C_3 e^{i\lambda_3 \frac{y}{a}} + C_4 e^{i\lambda_4 \frac{y}{a}} \right] \quad (70a)$$

With the roots λ_n the characteristic equation can also be written as

$$(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)(\lambda - \lambda_4) = 0$$

or resolved

$$\begin{aligned} \lambda^4 - \lambda^3(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4) + \lambda^2(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4) \\ - \lambda(\lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \lambda_1\lambda_3\lambda_4 + \lambda_2\lambda_3\lambda_4) + \lambda_1\lambda_2\lambda_3\lambda_4 = 0 \end{aligned} \quad (72)$$

A comparison of the coefficients of the term with λ^3 of equation (71) and (72) gives:

$$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = - \frac{{}^4D_{23}'}{D_{22}'} \kappa \quad (73)$$

The roots λ can, therefore, be expressed by

$$\left. \begin{aligned} \lambda_{1,2} &= \alpha \pm 3 - \frac{D_{23}'}{D_{22}'} \kappa \\ \lambda_{3,4} &= -\alpha \pm \gamma - \frac{D_{23}'}{D_{22}'} \kappa \end{aligned} \right\} \quad (74)$$

According to the principles governing the roots of equations with real coefficients, α must always be real and β and γ either real or purely imaginary.

Further comparison of the other coefficients in equations (71) and (72) and introduction of formula (74) gives in place of the characteristic equation the relations:

$$\begin{aligned}
 -2\alpha^2 - \beta^2 - \gamma^2 &= 2B_1^2 \kappa^2 - \frac{N_y a^2}{D_{22}} \\
 \alpha(\beta^2 - \gamma^2) &= 2B_2^3 \kappa^3 + \left(-\frac{N_y a^2}{D_{22}'} \frac{D_{23}'}{D_{22}'} + \frac{N_{xy} a^2}{D_{22}'} \right) \kappa
 \end{aligned} \tag{75}$$

$$(\alpha^2 - \beta^2)(\alpha^2 - \gamma^2) = B_3^4 \kappa^4 + \left[-\frac{N_y a^2}{D_{22}'} \left(\frac{D_{23}'}{D_{22}'} \right)^2 + 2 \frac{N_{xy} a^2}{D_{22}'} \frac{D_{23}'}{D_{22}'} - \frac{N_x a^2}{D_{22}'} \right] \kappa^2$$

The quantities B_1 , B_2 , and B_3 represent expressions of sums of the stiffness quantities of the plate:

$$\begin{aligned}
 2B_1^2 &= -6 \left(\frac{D_{23}'}{D_{22}'} \right)^2 + 2 \frac{D_{33}'}{D_{22}'} \\
 2B_2^3 &= -4 \left(\frac{D_{23}'}{D_{22}'} \right)^2 + 2 \frac{D_{23}'}{D_{22}'} \frac{D_{33}'}{D_{22}'} - 2 \frac{D_{13}'}{D_{22}'} \\
 B_3^4 &= -3 \left(\frac{D_{23}'}{D_{22}'} \right)^4 + 2 \left(\frac{D_{23}'}{D_{22}'} \right)^2 \frac{D_{33}'}{D_{22}'} - 4 \frac{D_{13}'}{D_{22}'} \frac{D_{23}'}{D_{22}'} + \frac{D_{11}'}{D_{22}'}
 \end{aligned} \tag{76}$$

3. The Boundary Conditions and Buckling Equations

As boundary condition for the plate strip, the two extreme cases of the clamped and freely supported edges are investigated.

In both cases, there is no displacement normal to the plate at the edges. For clamped support, the angle of inclination $\partial w / \partial y$ of the plate disappears at the edges, while, when freely supported, the fixed end moment M_y must disappear. The boundary conditions for fixed end support therefore read

$$w = 0 \quad \text{for} \quad y = \pm a \quad (\text{I})$$

$$\frac{\partial w}{\partial y} = 0 \quad \text{for} \quad y = \pm a$$

and for free support

$$w = 0 \quad \text{for} \quad y = \pm a \quad (\text{II})$$

$$M_y = 0 \quad \text{for} \quad y = \pm a$$

The four boundary conditions (I) and (II) give four linear homogeneous equations each, for the determination of the constants C_n in equation (70a) which, depending upon the boundary conditions, take different forms:

(α) for rigidly restrained support:

$$\text{for } w = 0, y = a: C_1 e^{i\lambda_1} + C_2 e^{i\lambda_2} + C_3 e^{i\lambda_3} + C_4 e^{i\lambda_4} = 0$$

$$\text{for } w = 0, y = -a: C_1 e^{-i\lambda_1} + C_2 e^{-i\lambda_2} + C_3 e^{-i\lambda_3} + C_4 e^{-i\lambda_4} = 0$$

$$\text{for } \frac{\partial w}{\partial y} = 0, y = a: C_1 \lambda_1 e^{i\lambda_1} + C_2 \lambda_2 e^{i\lambda_2} + C_3 \lambda_3 e^{i\lambda_3} + C_4 \lambda_4 e^{i\lambda_4} = 0$$

$$\text{for } \frac{\partial w}{\partial y} = 0, y = -a: C_1 \lambda_1 e^{-i\lambda_1} + C_2 \lambda_2 e^{-i\lambda_2} + C_3 \lambda_3 e^{-i\lambda_3} + C_4 \lambda_4 e^{-i\lambda_4} = 0 \quad (\text{I}(1))$$

Introducing the relation

$$e^{\pm i\lambda_n} = \cos \lambda_n \pm i \sin \lambda_n$$

and equating the real and imaginary parts of (I(1)) to zero results in

$$C_1 \cos \lambda_1 + C_2 \cos \lambda_2 + C_3 \cos \lambda_3 + C_4 \cos \lambda_4 = 0$$

$$C_1 \sin \lambda_1 + C_2 \sin \lambda_2 + C_3 \sin \lambda_3 + C_4 \sin \lambda_4 = 0$$

(I(2))

$$C_1 \lambda_1 \cos \lambda_1 + C_2 \lambda_2 \cos \lambda_2 + C_3 \lambda_3 \cos \lambda_3 + C_4 \lambda_4 \cos \lambda_4 = 0$$

$$C_1 \lambda_1 \sin \lambda_1 + C_2 \lambda_2 \sin \lambda_2 + C_3 \lambda_3 \sin \lambda_3 + C_4 \lambda_4 \sin \lambda_4 = 0$$

These equations have solutions different from zero only when the determinant of the coefficients disappears. The determinant put equal to zero gives a transcendental equation with the roots λ_n , the buckling equation.

This buckling equation reads therefore

$$\begin{vmatrix} \cos \lambda_1 & \cos \lambda_2 & \cos \lambda_3 & \cos \lambda_4 \\ \sin \lambda_1 & \sin \lambda_2 & \sin \lambda_3 & \sin \lambda_4 \\ \lambda_1 \cos \lambda_1 & \lambda_2 \cos \lambda_2 & \lambda_3 \cos \lambda_3 & \lambda_4 \cos \lambda_4 \\ \lambda_1 \sin \lambda_1 & \lambda_2 \sin \lambda_2 & \lambda_3 \sin \lambda_3 & \lambda_4 \sin \lambda_4 \end{vmatrix} = 0 \quad (\text{I(3)})$$

The solution of the determinant gives

$$(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4) \sin(\lambda_1 - \lambda_2) \sin(\lambda_3 - \lambda_4)$$

$$-(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_4) \sin(\lambda_1 - \lambda_3) \sin(\lambda_2 - \lambda_4) = 0 \quad (\text{I(4)})$$

Insertion of the expressions (74) for λ_n gives, after some rearrangements, the buckling equation for fixed end support in the form

$$2\beta\gamma(\cos 2\beta \cos 2\gamma - \cos 4\alpha) - (4\alpha^2 - \beta^2 - \gamma^2)\sin 2\beta \sin 2\gamma = 0 \quad (\text{Ia})$$

β . Free support

The first two equations for C_n (from $w = -0$, $y = \pm a$) agree with those applicable to fixed end support.

The third and fourth equations follow from the relation (cf. equation 28).

$$M_y = D_{12} \frac{\partial^2 w}{\partial x^2} + D_{22} \frac{\partial^2 w}{\partial y^2} + 2D_{23} \frac{\partial^2 w}{\partial xy} = 0 \quad \text{for } y = \pm a \quad (\text{II(1)})$$

The first term disappears, because $\frac{\partial^2 w}{\partial x^2} = -\frac{\kappa^2 w}{a^2} = 0$ for $y \pm a$.

With the relations from equation (70a)

$$\frac{\partial^2 w}{\partial y^2} = -e^{i\kappa \frac{x}{a}} \sum_{n=1}^4 C_n \frac{\lambda_n^2}{a^2} e^{i\lambda_n \frac{y}{a}}$$

and

$$\frac{\partial^2 w}{\partial xy} = -\frac{\kappa}{a} e^{i\kappa \frac{x}{a}} \sum_{n=1}^4 C_n \frac{\lambda_n}{a} e^{i\lambda_n \frac{y}{a}}$$

the equations II(1) give for $y = \pm a$ the two equations for the constants C_n

$$\sum_{n=1}^4 C_n e^{i\lambda_n} \left[D_{22}' \lambda_n^2 + 2D_{23}' \lambda_n \kappa \right] = 0$$

(II(2))

$$\sum_{n=1}^4 C_n e^{-i\lambda_n} \left[D_{22}' \lambda_n^2 + 2D_{23}' \lambda_n \kappa \right] = 0$$

Again introducing equation (77) in (II(2)) and putting the real and imaginary parts of the equation equal to zero gives the third and fourth equation for C_n in the form

$$\sum_{n=1}^4 C_n \cos \lambda_n \left[D_{22}' \lambda_n^2 + 2D_{23}' \lambda_n \kappa \right] = 0$$

$$\sum_{n=1}^4 C_n \sin \lambda_n \left[D_{22}' \lambda_n^2 + 2D_{23}' \lambda_n \kappa \right] = 0$$

(II(3))

Putting for simplification, $D_{22}' \lambda_n^2 + 2D_{23}' \lambda_n \kappa = D_n$ the four linear, homogeneous equations for C_n in the case of free support read

$$C_1 \cos \lambda_1 + C_2 \cos \lambda_2 + C_3 \cos \lambda_3 + C_4 \cos \lambda_4 = 0$$

$$C_1 \sin \lambda_1 + C_2 \sin \lambda_2 + C_3 \sin \lambda_3 + C_4 \sin \lambda_4 = 0$$

$$C_1 D_1 \cos \lambda_1 + C_2 D_2 \cos \lambda_2 + C_3 D_3 \cos \lambda_3 + C_4 D_4 \cos \lambda_4 = 0$$

$$C_1 D_1 \sin \lambda_1 + C_2 D_2 \sin \lambda_2 + C_3 D_3 \sin \lambda_3 + C_4 D_4 \sin \lambda_4 = 0$$

(II(4))

*The summation sign was omitted in these two equations in the original version of this paper.

The determinant of the coefficients of this equation system put equal to zero gives again the buckling equation for the case of free support. Development of the determinant results in

$$(D_1 - D_3)(D_2 - D_4)\sin(\lambda_1 - \lambda_2)\sin(\lambda_3 - \lambda_4) \\ -(D_1 - D_2)(D_3 - D_4)\sin(\lambda_1 - \lambda_3)\sin(\lambda_2 - \lambda_4) = 0 \quad (\text{II}(5))$$

By (74) the buckling equation for the freely supported plate follows after various rearrangements in the form

$$8\alpha^2\beta\gamma [\cos 2\beta \cos 2\gamma - \cos 4\alpha] \\ - \left[4\alpha^2(\beta^2 + \gamma^2) - (\beta^2 - \gamma^2)^2 \right] \sin 2\beta \sin 2\gamma = 0 \quad (\text{IIa})$$

It is interesting to note that rigidity value D_{23}'/D_{22}' characteristic of the general-orthotropic plate cancels out during the rearrangements, thus leaving for the general-orthotropic plate the same buckling equations from the boundary conditions, as obtained by Southwell-Skan for the isotropic plate.

4. Solution for Pure Axial Compression

For the case of pure axial compression, the equations (75) take the form

$$\begin{aligned} -2\alpha^2 - \beta^2 - \gamma^2 &= 2B_1^2\kappa^2 \\ \alpha(\beta^2 - \gamma^2) &= 2B_2^3\kappa^3 \\ (\alpha^2 - \beta^2)(\alpha^2 - \gamma^2) &= B_3^4\kappa^4 - n_x\kappa^2 \end{aligned} \quad (75a)$$

$N_x a^2/D_{22}$ being abbreviated to n_x .

The three equations (75a) in conjunction with the buckling equation (Ia) and (IIa) permit the four unknowns, α , β , γ , and n_x , to be defined for any, for the present assumed, fixed value of κ . The minimum value of the values of n_x with respect to κ gives the desired buckling load of the plate. The stiffness characteristics of the general-orthotropic plate enter through the constants B_1 , B_2 , and B_3 , which, in turn, are related to the quantities D_{11}' , D_{22}' , D_{33}' , D_{13}' , and D_{23}' in the qualifying equations (75a).

From the first two equations of (75a), β and γ are found as

$$\begin{aligned}\beta^2 &= -B_1^2 \kappa^2 + B_2^3 \frac{\kappa^3}{\alpha} - \alpha^2 \\ \gamma^2 &= -B_1^2 \kappa^2 - B_2^3 \frac{\kappa^3}{\alpha} - \alpha^2\end{aligned}\tag{78}$$

and from the third equation follows

$$n_x = \frac{N_x a^2}{D_{22}'} = B_3^4 \kappa^2 - \frac{(\alpha^2 - \beta^2)(\alpha^2 - \gamma^2)}{\kappa^2}\tag{79}$$

In the following β is always assumed real, while γ can become real as well as imaginary (reference 1, p. 280).

From the transcendental equations (Ia) and (IIa), which for imaginary γ can be written in the form

$$2\beta \frac{\gamma}{1} \left[\cos 2\beta \cosh 2\frac{\gamma}{1} - \cos 4\alpha \right] - \left[4\alpha^2 - \beta^2 - \gamma^2 \right] \sin 2\beta \sinh 2\frac{\gamma}{1} = 0 \tag{Ib}$$

and

$$\begin{aligned}8\alpha^2 \beta \frac{\gamma}{1} \left[\cos 2\beta \cosh 2\frac{\gamma}{1} - \cos 4\alpha \right] - \\ \left[4\alpha^2(\beta^2 + \gamma^2) - (\beta^2 - \gamma^2)^2 \right] \sin 2\beta \sinh 2\frac{\gamma}{1} = 0\end{aligned}\tag{IIb}$$

the value α for constant κ can be obtained from equation (78) and with it the quantity n_x from equation (79). By repeating the

calculation for other values of κ , n_x can be represented as function of κ , the lowest value of which is the looked-for buckling load n_x .

For the case of the special-orthotropic plate $\left[B_2 = 0, B_1^2 = \frac{D_{33}}{D_{22}}, \right.$ see equation (76)] a closed solution can be given for the buckling load of the freely supported plate. For the special-orthotropic plate, the equations (75a) assume the form

$$-2\alpha^2 - \beta^2 - \gamma^2 = 2B_1^2 \kappa^2 = 2 \frac{D_{33}}{D_{22}} \kappa^2$$

$$\alpha(\beta^2 - \gamma^2) = 0 \quad (75b)$$

$$(\alpha^2 - \beta^2)(\alpha^2 - \gamma^2) = B_3^4 \kappa^4 - n_x \kappa^2 = \frac{D_{11}}{D_{22}} \kappa^4 - n_x \kappa^2$$

From the second equation of (75b) follows $\alpha = 0$. The buckling equation reads then:

$$(\beta^2 + \gamma^2)^2 \sin 2\beta \sinh 2\frac{\gamma}{1} = 0 \quad (80)$$

Since $\sinh 2\frac{\gamma}{1} > 0$, the last equation gives

$$\sin 2\beta = 0 \quad \text{and} \quad \beta = \frac{\pi}{2}$$

From it, the first equation at (75a) follows as

$$\gamma^2 = -2 \frac{D_{23}}{D_{22}} \kappa^2 - \frac{\pi^2}{4}$$

and hence from (79)

$$n_x = \frac{N_x a^2}{D_{22}} = \frac{D_{11}}{D_{22}} \kappa^2 - \frac{\beta^2 \gamma^2}{\kappa^2} = \frac{D_{11} \kappa^2}{D_{22}} - \frac{\pi^2}{4 \kappa^2} \left(-\frac{2D_{33} \kappa^2}{D_{22}} - \frac{\pi^2}{4} \right)$$

$$\frac{N_x a^2}{D_{22}} = \frac{D_{11}}{D_{22}} \kappa^2 + \frac{1}{2} \pi^2 \frac{D_{33}}{D_{22}} + \frac{\pi^4}{16} \frac{1}{\kappa^2} \quad (81)$$

The value κ at which the minimum value of N_x appears follows from $\frac{\partial N_x}{\partial \kappa} = 0$ as

$$\kappa = \frac{\pi}{2} \sqrt[4]{\frac{D_{22}}{D_{11}}} \quad (82)$$

and, after introducing (82) in (81), the conventional buckling load formula (reference 6, p. 382).

$$N_x = \frac{\pi^2}{2a^2} \left(\sqrt{D_{11} D_{22}} + D_{33} \right)$$

and (83)

$$n_x = \frac{N_x a^2}{D_{22}} = \frac{\pi^2}{2} \left(\sqrt{\frac{D_{11}}{D_{22}}} + \frac{D_{33}}{D_{22}} \right)$$

The result indicates that the buckling load for pure compression of a special-orthotropic plate for $\omega = 0$ and for $\omega = 90^\circ$ is of the same magnitude, while the buckling lengths for $\omega = 0$ and $\omega = 90^\circ$ are of different magnitude.

The buckling load of the special-orthotropic plate for the case of clamped edges¹ cannot be given in closed form, but the solution can be represented in simple form as function of the "characteristic value" of the orthotropic plate $\beta = \frac{\sqrt{D_{11}D_{22}}}{D_{33}}$ and determined in simpler manner

than in the general-orthotropic case, because $\alpha = 0$.

The equations (78) hold for this case also, but the buckling equation takes the form

$$2\beta\frac{\gamma}{i}\left(1 - \cos 2\beta \cosh 2\frac{\gamma}{i}\right) = (\beta^2 + \gamma^2) \sin 2\beta \sinh 2\frac{\gamma}{i} \quad (84)$$

Rearrangement results in the transcendental equation

$$\beta \tan \beta = -\frac{\gamma}{i} \tanh \frac{\gamma}{i} \quad (85)$$

from equation (79) follows again

$$\frac{N_x a^2}{D_{22}} = \frac{D_{11}}{D_{22}} \kappa^2 - \gamma^2 \frac{\beta^2}{\kappa^2}$$

and from (75a)

$$\kappa^2 = \frac{\gamma^2 + \beta^2}{2 \frac{D_{33}}{D_{22}}}$$

hence the buckling load

$$N_x a^2 = \frac{D_{11}D_{22}}{2D_{22}} (\gamma^2 + \beta^2) + 2D_{33} \frac{\gamma^2 \beta^2}{\gamma^2 + \beta^2} \quad (86)$$

¹The investigation of the special-orthotropic plate with clamped edges does not appear in the literature and is therefore appended for the sake of completeness.

or

$$\frac{N_x a^2}{D_{33}} = - \frac{D_{11} D_{22}}{D_{33}^2} \frac{\gamma^2 + \beta^2}{2} + \frac{2\gamma^2 \beta^2}{\gamma^2 + \beta^2} \quad (86a)$$

The value $N_x a^2 / D_{33}$ for every fixed parameter $D_{11} D_{22} / D_{33}^2$ can be represented as function of the value γ regarded as independent variable (γ and β are connected through (85)).

The minimum of this function represents the looked-for buckling load. The result of this calculation is included in table 2 and plotted against the "plate value"

$$\frac{\sqrt{D_{11} D_{22}}}{D_{33}} = \vartheta$$

in figure 22. The closed solution for the case of the supported plate is included in the corresponding form in this diagram.

In Section IIIB the case of the clamped plate which permits no closed formal representation of the result is indicated by an approximation formula in the form

$$N_x = \frac{2\pi^2}{3a^2} \left(\sqrt{3D_{11} D_{22}} + D_{33} \right) \quad (87)$$

which is also included in figure 22 for comparison with the exact values. The accuracy of this formula can be substantially enhanced by writing (as seen from figure 22)

$$N_x = \frac{2\pi^2}{3a^2} \left(\sqrt{3D_{11} D_{22}} + 0.88D_{33} \right) \quad (87a)$$

The differences throughout the explored range $\vartheta = 0.2$ to 5 are very small, except in the range $\vartheta = 0$ to 0.2 where they become greater and at $\vartheta = 0$ assume a maximum of about 12 percent. At $\vartheta = 0.2$ the error is only 12 percent; at $\vartheta = 5$, about 1.8 percent. The values between these two characteristic values have an even smaller difference.

The case of the isotropic plate is contained in the special-orthotropic plate for the characteristic $\eta = 1$ and gives for the supported plate

$$N_x = \frac{\pi^2}{a^2} D = 9.87 \frac{D}{a^2}$$

and for the clamped plate

$$N_x = \frac{17.20D}{a^2}$$

5. Solution for Pure Shear

In the case of pure shear, the equations (75) read

$$-2\alpha^2 - \beta^2 - \gamma^2 = 2B_1^2 \kappa^2$$

$$\alpha(\beta^2 - \gamma^2) = 2B_2^3 \kappa^3 + n_{xy} \kappa \quad (75c)$$

$$(\alpha^2 - \beta^2)(\alpha^2 - \gamma^2) = B_3^4 \kappa^4 + 2n_{xy} A \kappa^2$$

when writing for abbreviation, $\frac{N_{xy} a^2}{D_{22}} = n_{xy}$ and $A = \frac{D_{23}'}{D_{22}'}$.

As in the case of pure compression, the unknown α , β , and γ can be determined from the three equations and the buckling equation (I) and (II), when an arbitrary value is assumed for κ . After determining several κ values, the minimum of the function $n_{xy} = f(\kappa)$ gives the desired buckling load of the general-orthotropic plate. From

equations (75) follow after considerable calculations the quantities β and γ

$$\begin{aligned}\beta^2 &= -B_1^2 \kappa^2 - \alpha^2 - 2A\alpha\kappa + \sqrt{4A^2 \kappa^2 \alpha^2 + (4AB_2^3 - B_3^4) \kappa^4 + (B_1^2 \kappa^2 + 2\alpha^2)^2} \\ \gamma^2 &= -B_1^2 \kappa^2 - \alpha^2 + 2A\alpha\kappa - \sqrt{4A^2 \kappa^2 \alpha^2 + (4AB_2^3 - B_3^4) \kappa^4 + (B_1^2 \kappa^2 + 2\alpha^2)^2}\end{aligned}\quad (88)$$

and, in addition

$$n_{xy} = \frac{N_{xy} a^2}{D_{22}} = \frac{\alpha}{\kappa} (\beta^2 - \gamma^2) - 2B_2^3 \kappa^3 \quad (89)$$

From the buckling equations (I) and (II), the value α and hence n_{xy} is obtained again for chosen values of κ with the aid of the equations. The minimum value of n_{xy} with respect to κ represents the buckling load.

For the case of the special-orthotropic plate the equations (75b) simplify to

$$-2\alpha^2 - \beta^2 - \gamma^2 = 2 \frac{D_{33}}{D_{22}} \kappa^2$$

$$\alpha(\beta^2 - \gamma^2) = n_{xy} \kappa \quad (75d)$$

$$(\alpha^2 - \beta^2)(\alpha^2 - \gamma^2) = \frac{D_{11}}{D_{22}} \kappa^4$$

because $B_1^2 = \frac{D_{33}}{D_{22}}$, $B_2^3 = 0$, $B_3^4 = \frac{D_{11}}{D_{22}}$, and $A = 0$.

The buckling equations are also applicable to this case without change. Seydel (reference 3) determined the buckling loads for this

case in a very thorough study by means of equations (75d) and plotted the result against the plate characteristic.

For the case of the isotropic plate with the still further simplified equations,

$$\begin{aligned} -2\alpha^2 - \beta^2 - \gamma^2 &= 2\kappa^2 \\ \alpha(\beta^2 - \gamma^2) &= n_{xy}\kappa \\ (\alpha^2 - \beta^2)(\alpha^2 - \gamma^2) &= \kappa^4 \end{aligned} \quad (75e)$$

the solution by Southwell and Skan (reference 4), was obtained with $n_{xy} = \frac{N_{xy}a^2}{D} = 13.16$ for the supported plate and $n_{xy} = 22.70$ for the clamped plate.

B. THE APPROXIMATE SOLUTION OF THE PROBLEM

An increase in the number of elastic constants on the general-orthotropic plate compared to the general-orthotropic plate (in pure compression the number of elastic constants increases from two to three and in pure shear from two to four (see equations (75a) and (75b) or (75c) and (75d)) renders the determination of the buckling loads and buckling lengths by means of the previously discussed exact method extremely time-consuming, since in view of the greater combination possibilities of the elastic constants a much greater number of buckling load determinations must be carried out.

1. Method of Solution

The approximate solution can be obtained by the energy method. On transition of the plate from the plane to the buckled equilibrium position, hence, on reaching the critical buckling load, the energy stored in the plate by the bending strain is exactly equal to the work performed by the external forces:

$$A_I - A_A = 0 \quad (90)$$

The internal bending energy owing to the deformation w of the plate is indicated by the equations (36); the energy of the external forces during the buckling process may be expressed in the following form

$$A_A = -\frac{1}{2} \iint \left[N_x \left(\frac{\partial w}{\partial x} \right)^2 + 2N_{xy} \left(\frac{\partial w}{\partial x} \right) \left(\frac{\partial w}{\partial y} \right) + N_y \left(\frac{\partial w}{\partial y} \right)^2 \right] dx \, dy \quad (91)$$

Equation 10 becomes then

$$\begin{aligned} \frac{s^3}{12} \iint & \left[\alpha_{11} \left(\frac{\partial^2 w}{\partial x^2} \right)^2 + 2\alpha_{12} \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} + \alpha_{22} \left(\frac{\partial^2 w}{\partial y^2} \right)^2 + 4\alpha_{13} \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial x \partial y} \right. \\ & + 4\alpha_{23} \frac{\partial^2 w}{\partial y^2} \frac{\partial^2 w}{\partial x \partial y} + 4\alpha_{33} \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 \Big] dx dy - \iint \left[N_x \left(\frac{\partial w}{\partial x} \right)^2 \right. \\ & + 2N_{xy} \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} + N_y \left(\frac{\partial w}{\partial y} \right)^2 \Big] dx dy \end{aligned} \quad (92)$$

The condition for stable equilibrium of the plate that the energy change during the buckling process shall be a minimum, hence

$$\delta A - \delta(A_I - A_A)$$

can be satisfied by variation of the displacement w of the plate in equation (12). This variation process yields the differential equation (cf. III, A, 1) for the elastic surface of the buckled plate and the characteristic values of this differential equation give the exact buckling load. But if the buckling load is to be determined only approximately, an approximation formula for the buckling form can be introduced in equation (12), which differs from the actual form of the buckling surface. In this case, a formula is chosen which contains two free values which, by application of the condition of minimum energy change during the buckling process, can be so determined that the presumed buckling form is as close as possible to the actual form.

A simple formula is applied in the form

$$w = A \cos \frac{\pi y}{b} \sin \frac{\pi}{L} (x - \epsilon y) \quad (93)$$

for the freely supported longitudinal edges of the plate, and

$$w = \frac{A}{2} \left(1 - A \cos \frac{2\pi y}{b} \right) \sin \frac{\pi}{L} (x - \epsilon y) \quad (94)$$

for the fixed end support of the longitudinal edges.

This function, originally given by Timoshenko (reference 6, p. 365), represents infinitely long buckled surfaces with the half-wave length L (fig. 24), whose straight nodal lines running diagonally to the longitudinal axis x form with the y axis the angle ϕ , the tangent of which is equal to ϵ ; L and ϵ are the free values.

Formula (93) satisfies only the geometric boundary condition $w = 0$, while the boundary condition $M_y = 0$ (see equation II(1)) is not satisfied, because neither $\partial^2 w / \partial y^2$ nor $\partial^2 w / \partial x \partial y$ disappear at the boundaries $y = \pm a$, as a rule.

On the other hand, formula (94) satisfies both $w = 0$ and $\frac{\partial w}{\partial y} = 0$, but the two formulas still do not satisfy the differential equation of the problem.

The nodal lines of the actual buckling surfaces are, in general, not straight lines. But, as even comparatively rough assumptions for the buckling form yield satisfactory approximate solutions for the buckling load, the simple formulas (93) and (94) should afford sufficiently exact buckling loads.

As in the exact solution of the differential equation, so the buckling load of an infinitely long orthotropic plate strip freely supported, or clamped, respectively, will be investigated for pure axial compression and pure shear.

2. Approximate Solution for Pure Axial Compression

(a) Free Support

Performing the integrations indicated in the equation (92) by means of formula (93), the buckling load for the case of pure axial compression follows from (92) as

$$N_x = \frac{\pi^2}{b^2} \left[D_{11} \cdot \frac{b^2}{L^2} + 2D_{33} \cdot \left(\frac{b^2}{L^2} \epsilon^2 + 1 \right) + D_{22} \cdot \left(6\epsilon^2 + \frac{b^2}{L^2} \epsilon^4 + \frac{L^2}{b^2} \right) - 4D_{13} \cdot \epsilon \frac{b^2}{L^2} - 4D_{23} \cdot \left(\frac{b^2}{L^2} \epsilon^3 + 3\epsilon \right) \right] \quad (95)$$

The two free values L and α are obtained from the minimum conditions $\frac{\partial A}{\partial L} = 0$ and $\frac{\partial A}{\partial \epsilon} = 0$, or, what amounts to the same thing, from the conditions $\frac{\partial N_x}{\partial L} = 0$ and $\frac{\partial N_x}{\partial \epsilon} = 0$.

From $\frac{\partial N_x}{\partial L} = 0$ follows

$$\frac{L^4}{b^4} = \epsilon^4 - 4 \frac{D_{23}'}{D_{22}'} \epsilon^3 + 2 \frac{D_{33}'}{D_{22}'} \epsilon^2 - 4 \frac{D_{13}'}{D_{22}'} \epsilon + \frac{D_{11}'}{D_{22}'} \quad (96)$$

from $\frac{\partial N_x}{\partial \epsilon} = 0$ follows

$$\frac{b^2}{L^2} \left(\epsilon^3 - 3 \frac{D_{23}'}{D_{22}'} \epsilon^2 - \frac{D_{33}'}{D_{22}'} \epsilon - \frac{D_{13}'}{D_{22}'} \right) - 3 \left(\frac{D_{23}'}{D_{22}'} - \epsilon \right) = 0 \quad (97)$$

Although one unknown L/b can be eliminated from the equations (96) and (97), no closed expression for ϵ with respect to the stiffness quantities can be given, on account of the ensuing equation of the sixth degree for ϵ ; the values ϵ and L/b from equations (96) and (97) must therefore be obtained by a graphical method.

The introduction of L^4/b^4 in N_x simplifies the expression for N_x in

$$N_x = 2 \frac{\pi^2}{b^2} D_{22}' \left(\frac{L^2}{b^2} + 3\epsilon^2 + \frac{D_{33}'}{D_{22}'} - 6 \frac{D_{23}'}{D_{22}'} \epsilon \right) \quad (98)$$

For the special-orthotropic plate, formulas (96), (97), and (98) give the conventional closed relation for the buckling load of an orthotropic plate. The terms with D_{13}' and D_{23}' disappear in equation (97), hence $\epsilon = 0$ and equation (96) therefore gives

$$\frac{L}{b} = \sqrt[4]{\frac{D_{11}'}{D_{22}'}} \quad (99)$$

For N_x , after insertion of (99)

$$N_x = \frac{2\pi^2}{b^2} (D_{11}D_{22} + D_{33}) \quad (100)$$

This relation applies exact, since for $\epsilon = 0$, and $D_{13}' = D_{23}' = 0$, the formula $w = A \cos \frac{\pi y}{b} \sin \frac{\pi x}{L}$ satisfies the differential equation of the special-orthotropic plate (equation 32).

A simple relation holds further for the particular case of an orthotropic plate of equal principal stiffnesses $D_{11}' = D_{22}'$, the directions of the fibers of which run at 0° and 45° to the edges, since in these cases the terms with D_{13}' and D_{23}' also disappear (cf. equation (23a)) and $\epsilon = 0$. The stiffnesses $D_{11}' = D_{22}'$ and D_{33}' refer then, of course, to the momentary direction of the axes.

The buckling load is then

$$N_x = \frac{2\pi^2}{b^2} (D_{22}' + D_{33}') \quad (101)$$

and the half-wave length $\frac{L}{b} = 1$.

This simple relation does not hold, however, when arranging an orthotropic plate of equal principal stiffnesses under arbitrary direction of fibers because then ϵ does not disappear.

Lastly, for the isotropic plate, it gives

$$D_{11}' = D_{22}' = D_{33}' = D$$

and

$$D_{13}' = D_{23}' = 0$$

and hence

$$N_x = \frac{4\pi^2}{b^2} D$$

(b) Clamped Support

The determination of the corresponding buckling loads of the plate with clamped edges in axial compression by formula (94) gives

$$N_x = \frac{\pi^2}{b^2} \left[D_{11}' \frac{b^2}{L^2} - 4D_{13}' \frac{b^2}{L^2} \epsilon + 2D_{33}' \left(\frac{b^2}{L^2} \epsilon^2 + \frac{4}{3} \right) - 4D_{23}' \left(\frac{b^2}{L^2} \epsilon^3 + 4\epsilon \right) + D_{22}' \left(\frac{b^2}{L^2} \epsilon^4 + 8\epsilon^2 + \frac{16}{3} \frac{L^2}{b^2} \right) \right] \quad (102)$$

The minimum conditions $\frac{\partial N_x}{\partial L} = 0$ and $\frac{\partial N_x}{\partial \epsilon} = 0$ give the two equations for defining the free values ϵ and L/b :

$$\frac{16}{3} \frac{L^4}{b^4} = \epsilon^4 - 4 \frac{D_{23}'}{D_{22}'} \epsilon^3 + 2 \frac{D_{33}'}{D_{22}'} \epsilon^2 - 4 \frac{D_{13}'}{D_{22}'} \epsilon + \frac{D_{11}'}{D_{22}'} \quad (103)$$

$$\frac{1}{4} \frac{b^2}{L^2} \left(\epsilon^3 - 3 \frac{D_{23}'}{D_{22}'} \epsilon^2 + \frac{D_{33}'}{D_{22}'} \epsilon - \frac{D_{13}'}{D_{22}'} \right) - \left(\frac{D_{23}'}{D_{22}'} - \epsilon \right) = 0 \quad (104)$$

On introducing L^4/b^4 into the equation for N_x , the simplified expression

$$N_x = \frac{8}{3} \frac{\pi^2}{b^2} D_{22}' \left(4 \frac{L^2}{b^2} + 3\epsilon^2 + \frac{D_{33}'}{D_{22}'} - 6 \frac{D_{23}'}{D_{22}'} \epsilon \right) \quad (105)$$

is obtained.

For the case of the special-orthotropic plate, equation (103) gives

$$\frac{L}{b} = \frac{1}{2} \sqrt[4]{3 \frac{D_{11}}{D_{22}}} = \frac{1.316}{2} \sqrt[4]{\frac{D_{11}}{D_{22}}} = 0.658 \sqrt[4]{\frac{D_{11}}{D_{22}}} \quad (106)$$

The buckling load follows from (105) as

$$N_x = \frac{8}{3} \frac{\pi^2}{b^2} \left(\sqrt{3D_{11}D_{22}} + D_{33} \right) \quad (107)$$

while the expression applies exact for the buckling load of the supported orthotropic plate, the expression (107) for the buckling load of the clamped plate is only approximately valid, because the formula (97) for $\epsilon = 0$ does not satisfy the differential equation.

The relation for the buckling load in the particular case of equal principal stiffnesses ($\omega = 0^\circ$ and 45°) of the clamped plates reads

$$N_x = \frac{8}{3} \frac{\pi^2}{b^2} \left(\sqrt{3}D_{11} + D_{33} \right)$$

and the half-wave length

$$\frac{L}{b} = \frac{1}{2} \sqrt[4]{3} = 0.658 \quad (108)$$

The approximate expression for the isotropic restrained plate is

$$N_x = \frac{8(1 + \sqrt{3})}{3} \frac{\pi^2}{b^2} D = \frac{8 \times 2.731}{3} \frac{\pi^2}{b^2} D = 7.28 \frac{\pi^2}{b^2} D$$

The exact solution is (reference 6, p. 345)

$$N_x = 6.96 \frac{\pi^2}{b^2} D$$

3. Approximate Solution for Pure Shear

(a) Free Support

As for the pure compression the integrations together with formula (93) give the buckling load

$$N_{xy} = \frac{\pi^2}{2\epsilon b^2} \left[D_{11}' \frac{b^2}{L^2} + 2D_{33}' \left(\frac{b^2}{L^2} \epsilon^2 + 1 \right) + D_{22}' \left(6\epsilon^2 + \frac{b^2}{L^2} \epsilon^4 + \frac{L^2}{b^2} \right) - 4D_{13}' \epsilon \frac{b^2}{L^2} - 4D_{23}' \left(\frac{b^2}{L^2} \epsilon^3 + 3\epsilon \right) \right] \quad (109)$$

The expression for the buckling load in shear therefore agrees, up to the factor $1/2\epsilon$, with the relation for the buckling load in compression. Consequently, the equation $\partial N_{xy} / \partial L \epsilon$ for determining L and ϵ must be identical with the corresponding equation (96).

The second qualifying equation $\partial N_{xy} / \partial \epsilon$ gives the relation

$$\frac{b^2}{L^2} = \left(3\epsilon^4 - 8 \frac{D_{23}'}{D_{22}'} \epsilon^3 + 2 \frac{D_{33}'}{D_{22}'} \epsilon^2 - \frac{D_{11}'}{D_{22}'} \right) + 6\epsilon^2 - 2 \frac{D_{33}'}{D_{22}'} - \frac{L^2}{b^2} = 0 \quad (110)$$

Introducing equation (96) in (109) again gives the simplified expression for N_{xy}

$$N_{xy} = \frac{\pi^2}{\epsilon b^2} D_{22}' \left(\frac{L^2}{b^2} + 3\epsilon^2 + \frac{D_{33}'}{D_{22}'} - 6 \frac{D_{23}'}{D_{22}'} \epsilon \right) \quad (111)$$

For the isotropic plate, $D_{22} = D_{33} = D$; $D_{23} = 0$; hence the expression

$$N_{xy} = \frac{\pi^2}{\epsilon b^2} D \left(3\epsilon^2 + \frac{L^2}{b^2} + 1 \right) \quad (112)$$

This equation is identical with Timoshenko's result (reference 6, p. 361)

$$\epsilon = \frac{1}{2} \sqrt{2} = 0.706$$

and

$$\frac{L}{b} = \sqrt{1 + \alpha^2} = 1.225$$

hence by (112)

$$N_{xy} = \frac{8}{\sqrt{2}} \frac{\pi^2}{b^2} D = 5.66 \frac{\pi^2}{b^2} D$$

The exact solution is

$$N_{xy} = 5.35 \frac{\pi^2}{b^2} D,$$

the error of the approximate solution amounts to about 6 percent.

(b) Clamped Support

The corresponding calculation for the clamped plate gives the qualifying equations for ϵ and L ; in this instance, equation (103) is applicable as in pure axial compression and also the equation

$$\frac{b^2}{L^2} \left(3\epsilon^4 - 8 \frac{D_{23}'}{D_{22}'} \epsilon^3 + 2 \frac{D_{33}'}{D_{22}'} \epsilon^2 - \frac{D_{11}'}{D_{22}'} \right) + 8\epsilon^2 - \frac{8}{3} \frac{D_{33}'}{D_{22}'} - \frac{16L^2}{3b^2} = 0 \quad (112)$$

The critical shear load is

$$N_{xy} = \frac{4}{3} \frac{\pi^2}{\epsilon b^2} D_{22}' \left(4 \frac{L^2}{b^2} + 3\epsilon^2 + \frac{D_{33}'}{D_{22}'} - 6 \frac{D_{23}'}{D_{22}'} \epsilon \right) \quad (113)$$

For the isotropic plate, equation (103) and (112) give $\epsilon = 0.760$ and $\frac{I}{b} = 0.826$ and hence by (113)

$$N_x = 9.60 \frac{\pi^2 D}{b^2}$$

Since the exact solution gives

$$N_x = 8.98 \frac{\pi^2 D}{b^2}$$

the error of the approximate solution amounts to 6.7 percent.

C. BUCKLING LOAD OF THE ORTHOTROPIC PLATE UNDER COMPRESSION AND

SHEAR LOADING FOR THE PARTICULAR CASE $\omega = 45^\circ$

In order to minimize the paper work required for the determination of the buckling loads of the general-orthotropic plate, the results for several important specific cases are reproduced.

Of particular interest in practical application is the case of the general-orthotropic plate where the directions of the principal axes of the plate are inclined at a 45° angle relative to the plate edges. For the calculation and especially for the representation of the results, this case yields simplifications, since at $\omega = 45^\circ$ the relations $D_{11}' = D_{22}'$ are applicable (see equation (23a)). The results can then be represented in general form as function of the stiffness ratios D_{23}'/D_{22}' and D_{33}'/D_{22}' , since, owing to equation (23a), only these two values appear in the qualifying equations ((75) and (76)). These quantities are determined in the subsequent section (IIID) for the special stiffness combinations of plywood of varying construction over the entire range of angles ω ($\omega = 0$ to 180°) in order to demonstrate the characteristic relationship between buckling loads, buckling lengths, and angle ω .

1. Results for Pure Compression

The buckling loads n_x of the infinitely long general-orthotropic plate strip freely supported at the longitudinal edges and stressed in pure compression are shown plotted against the parameters D_{23}'/D_{22}' and

D_{33}'/D_{22}' in figure 25 for the specific case $\omega = 45^\circ$. The values were obtained by the approximation method described in section IIIB. The approximation is made by the graphical determination of the values L/b and ϵ from equations (96) and (97) and their subsequent insertion in equation (98).

For purposes of comparison with the results obtained by the exact calculation a few of the n_x values obtained by the latter method have been included in figure 25.

The method of determining the exact buckling loads is as follows: first, the constants B_1 , B_2 , and B_3 for a specific pair of values D_{23}'/D_{22}' , D_{33}'/D_{22}' are determined by equation (76). After these constants are introduced in equation (76a), the buckling load n_x for a number of specifically assumed values of κ can be obtained with the aid of equation (78) and the buckling equation (IIb) from equation (79). The characteristic behavior of the function $n_x = f(\kappa)$ is shown in figure 23 (for the pair $\frac{D_{23}'}{D_{22}'} = 0.5$, $\frac{D_{33}'}{D_{22}'} = 2$). The minimum of this function represents the looked-for buckling load.

A number of further exact results are obtainable from the qualifying equations (75a), without making the previously described calculation, for various particular combinations of D_{23}'/D_{22}' and D_{33}'/D_{22}' :

(a) When the stiffness ratio D_{23}'/D_{22}' takes the value zero, that is, when the plate has the same modulus of elasticity ($E_{11} = E_{22}$) along both principal axes, the qualifying equations (75a) can be written in the form

$$-2\alpha^2 - \beta^2 - \gamma^2 = 2 \frac{D_{33}'}{D_{22}'} \kappa^2$$

$$\alpha(\beta^2 - \gamma^2) = 0 \quad (75b)$$

$$(\alpha^2 - \beta^2)(\alpha^2 - \gamma^2) = \frac{D_{11}'}{D_{22}'} \kappa^4 - n_x \kappa^2 = \kappa^4 - n_x \kappa^2$$

The equations agree with the equations (75b) for the special orthotropic plate, except that instead of the stiffness ratios D_{33}/D_{22} and D_{11}/D_{22} referred to the principal axes directions, the ratios D_{33}'/D_{22}' and D_{11}'/D_{22}' appear for $\omega = 45^\circ$. At $\omega = 45^\circ$, D_{11}'/D_{22}' has the value unity.

The results obtained for the special-orthotropic plate apply also when $\frac{D_{23}'}{D_{22}'} = 0$, to the case $\omega = 45^\circ$. The plate value $\frac{D_{33}}{\sqrt{D_{11}D_{22}}}$ becomes then the expression D_{33}'/D_{22}' . The buckling loads represented in figure 25 for $\frac{D_{23}'}{D_{22}'} = 0$ by a straight line are therefore exact values.

(b) A number of other exact values may be found in simple manner for the particular case

$$B_2^3 = \frac{D_{23}'}{D_{22}'} \left[-2 \left(\frac{D_{33}'}{D_{22}'} \right)^2 + \frac{D_{33}'}{D_{22}'} - 1 \right] = 0 \quad (114)$$

The qualifying equations take then the form

$$-2\alpha^2 - \beta^2 - \gamma^2 = 2B_1^2 \kappa^2$$

$$\alpha(\beta^2 - \gamma^2) = 0 \quad (115)$$

$$(\alpha^2 - \beta^2)(\alpha^2 - \gamma^2) = B_3^4 \kappa^4 - N_x \kappa^2$$

For the special-orthotropic plate (the buckling equation (Ia) holds in the same form for the general as well as for the special-orthotropic plate), the exact relation

$$n_x = \frac{N_x \alpha^2}{D_{22}'} = \frac{\pi^2}{2} (B_1^2 + B_3^2) \quad (116)$$

is obtained, since $\alpha = 0$.

The wave length is

$$\kappa = \frac{\pi}{2} \frac{1}{B_3}$$

and

(117)

$$\frac{L}{b} = B_3$$

For this particular case between D_{23}'/D_{22}' and D_{33}'/D_{22}' , the relation

$$2 \left(\frac{D_{23}'}{D_{22}'} \right)^2 = \frac{D_{33}'}{D_{22}'} - 1 \quad (114a)$$

is defined by equation (114).

Hence by equations (76)

$$B_1^2 = B_3^2 = \frac{1}{2} \left(3 - \frac{D_{33}'}{D_{22}'} \right) = 1 - \left(\frac{D_{23}'}{D_{22}'} \right)^2 \quad (118)$$

Insertion of these values in (116) gives for the buckling load

$$n_x = \frac{\pi^2}{2} \left(3 - \frac{D_{33}'}{D_{22}'} \right) = \pi^2 \left[1 - \left(\frac{D_{23}'}{D_{22}'} \right)^2 \right] \quad (119)$$

and for the buckling length

$$\frac{L}{b} = B_3 = \left[\frac{1}{2} \left(3 - \frac{D_{33}'}{D_{22}'} \right) \right]^{1/2} = \left[1 - \left(\frac{D_{23}'}{D_{22}'} \right)^2 \right]^{1/2} \quad (120)$$

The equation (119) represents a number of exact buckling loads, which in figure 25 lie again on a straight line (denoted with subscript $B_2 = 0$). In the particular case $\frac{D_{23}'}{D_{22}'} = 0$ and $B_3 = 0$, the values $n_x = \pi^2$ and $\frac{L}{b} = 1$ valid for the isotropic plate are obtained for $\frac{D_{33}'}{D_{22}'} = 0$.

(c) For the comparison of the exact with the approximate solution, it is further of interest that the approximation formula w (equation 93) for the buckling surface for the two particular cases treated in (a) and (b) satisfy the boundary conditions of the problem and yields the buckling loads which agree with the exact values. Accordingly it may be presumed that the approximated results in the vicinity of these particular cases themselves do not differ very much from the exact values.

Introducing the approximation formula for w

$$w = A \sin \frac{\pi y}{b} \sin \frac{\pi}{L} (x - \epsilon y) \quad (93)$$

in the boundary conditions for the supported plate

$$(1) \quad w = 0 \quad \text{for} \quad y = \frac{b}{2} \quad \text{and} \quad y = -\frac{b}{2}$$

$$(2) \quad M_y = D_{12}' \frac{\partial^2 w}{\partial x^2} + D_{22}' \frac{\partial^2 w}{\partial y^2} + 2D_{23}' \frac{\partial^2 w}{\partial x \partial y} = 0 \quad (121)$$

$$\text{for} \quad y = \frac{b}{2} \quad \text{and} \quad y = -\frac{b}{2}$$

it follows

$$(1) \quad \text{that} \quad w = 0 \quad \text{is always satisfied for} \quad y = \frac{b}{2} \quad \text{and} \quad y = -\frac{b}{2}$$

$$(2) \quad \text{that} \quad \frac{\partial^2 w}{\partial y^2} \quad \text{disappears for} \quad y = \frac{b}{2} \quad \text{and} \quad y = -\frac{b}{2}, \quad \text{leaving as} \quad (122)$$

condition

$$D_{22}' \frac{\partial^2 w}{\partial y^2} + 2D_{23}' \frac{\partial^2 w}{\partial x y} = 0 \quad \text{for} \quad y = \frac{b}{2} \quad \text{and} \quad y = -\frac{b}{2}$$

After inserting $\partial^2 w / \partial y^2$ and $\partial^2 w / \partial x \partial y$, it is apparent that equation (112) can be satisfied only when the relation

$$\epsilon = \frac{D_{23}'}{D_{22}'} \quad (123)$$

is valid.

The case $\epsilon = \frac{D_{23}'}{D_{22}'} = 0$ yields from the approximation equation the buckling load

$$n_x = \frac{\pi^2}{b} \left(1 + \frac{D_{33}'}{D_{22}'} \right) \quad (101)$$

already derived in section B, II, 2 and which agrees with the exact result.

The relation $\epsilon = \frac{D_{23}'}{D_{22}'}$ introduced in the approximate equation (96) gives the buckling lengths

$$\frac{L^2}{b^2} = \frac{1}{2} \left(3 - \frac{D_{33}'}{D_{22}'} \right) \quad (120a)$$

From equation (97), further follows the relation

$$2 \left(\frac{D_{23}'}{D_{22}'} \right)^2 = \frac{D_{33}'}{D_{22}'} + 1 = 0 \quad (114b)$$

and from equation (98) the buckling load

$$n_x = \frac{\pi^2}{2} \left(3 - \frac{D_{33}'}{D_{22}'} \right) \quad (119a)$$

The expressions for L^2/b^2 and n_x obtained from the approximate equations with the condition $\epsilon = \frac{D_{23}'}{D_{22}'}$ are therefore in agreement with the exact values for the specific case $B_2 = 0$. That this actually is the case follows from the equation (114b) which is identical with the qualifying equation $B_2 = 0$.

These considerations indicate that for $B_2 = 0$ the approximate solution agrees with the exact solution as obtained by the differential equation. The nodal lines of the buckling surface are actually straight lines - which is not the case as a rule - running at an angle with respect to the axis, and the tangent of which is exactly equal to D_{23}'/D_{22}' .

In figure 26, the buckling loads n_x are shown plotted against D_{23}'/D_{22}' as abscissa and D_{33}'/D_{22}' as parameter. It is seen that the buckling load is maximum for $\frac{D_{23}'}{D_{22}'} = 0$, that is, for plates of equal stiffness in principal axes directions (cf. equation (24b)), and that the buckling load decreases with increasing (absolute) value D_{23}'/D_{22}' , that is, increasing difference in stiffness in the principal axes directions of the plate, and assumes the value zero in the case of vanishing flexural stiffness in one principal axis direction $\left(\frac{D_{23}'}{D_{22}'} = \pm 1\right)$.

The buckling loads lie within an area defined by the range of existence of the factors D_{23}'/D_{22}' and D_{33}'/D_{22}' (fig. 5a). The boundary lines of the existence range shown in figures 25 and 26 represent the boundary lines traced in these diagrams. Since the compressive load is symmetrical with the longitudinal axis of the plate, the buckling loads for $\omega = 45^\circ \left(\frac{D_{23}'}{D_{22}'} > 0\right)$ and $\omega = 135^\circ \left(\frac{D_{23}'}{D_{22}'} < 0\right)$ are the same.

Therefore, the two boundary lines $\frac{D_{23}'}{D_{22}'} = \frac{1}{4} \left(1 + \frac{D_{33}'}{D_{22}'}\right)$ and

$\frac{D_{23}'}{D_{22}'} = -\frac{1}{4} \left(1 + \frac{D_{33}'}{D_{22}'}\right)$ merge into one line.

The half-wave length L/b , that is, the spacing of the nodal lines of the elastic surface of the buckled plate, was also computed from equation (96) by the approximate method and plotted against D_{23}'/D_{22}' and D_{33}'/D_{22}' in figure 27.

The exact value of the half-wave lengths corresponds to the value κ obtained in the determination of the buckling loads for which n_x becomes a minimum (fig. 23). The connection between L/b and κ is apparent from the equation (70) for the exact formula of the elastic surface;

since $e^{i\kappa \frac{x}{a}}$ is a periodic function, the half wave L must be equal to half the period of the function, hence

$$\kappa \frac{L}{a} = \pi, \text{ and } \frac{L}{b} = \frac{2\pi}{\kappa} \quad (124)$$

The half-wave lengths for $\frac{D_{23}'}{D_{22}'} = 0$ and $B_2 = 0$ in figure 27 again represent exact values. They indicate that the error introduced by the approximate solution is rather small and exceeds no more than 1 percent in the analyzed cases.

Figure 27 indicates that for $\frac{D_{23}'}{D_{22}'} = 0$, that is, on plates with equal flexural stiffness in both principal axes directions the half-wave length is $\frac{L}{b} = 1$, while with increasing D_{23}'/D_{22}' , that is, growing difference in principal flexural stiffnesses, L/b decreases, and that in the extreme case of vanishing flexural stiffness along one principal axis $\frac{D_{23}'}{D_{22}'} = \pm 1$, L/b drops to zero. The boundary lines of the existence range are also shown.

The value ϵ , that is, the tangent to the angle formed by the straight nodal lines of the approximated buckling surface of the plate running obliquely to the x axis with the y axis, can be taken from figure 28. The straight line according to the approximation equation (93) reproduces in general, the actually forming nodal line only approximately.

It is only in the cases $\frac{D_{23}'}{D_{22}'} = 0$ and $B_2 = 0$ that the nodal lines at

angle $\epsilon = 0$ and $\epsilon = \frac{D_{23}'}{D_{22}'}$ are actually straight lines, because in these instances the approximation formula for the elastic surface is exactly correct

Figure 28 indicates that the value ϵ becomes zero also for $\frac{D_{23}'}{D_{22}'} = 0$, that is, the nodal lines are at right angles to the edges, as

is readily apparent in view of the symmetrical structure of the plate. In all other cases, the nodal lines run obliquely to the edges even under pure compression, similar to the case of the buckled isotropic plate loaded in shear. As D_{23}'/D_{22}' increases, the value ϵ increases, to reach in the extreme case of $\frac{D_{23}'}{D_{22}'} = \pm 1$ the value unity, the slope of the nodal lines to reach the value 45° . The boundary lines of the existence range are included.

2. Results for Pure Shear

The buckling loads n_{xy} of the infinitely long orthotropic plate strip, freely supported at the longitudinal edges and stressed in pure shear, are represented in figures 29 and 30 for $\omega = 45^\circ$ and $\omega = 135^\circ$ plotted against the stiffness ratios D_{23}'/D_{22}' and D_{33}'/D_{22}' . The results were obtained by the approximate method.

To estimate the error of the approximate relative to the exact calculation, the buckling loads for a number of stiffness combinations were computed by the exact method and plotted in figure 29a. In table 3, the exact result is compared with the approximate values. The comparison indicates that the result of the approximate calculation, especially at higher values of D_{23}'/D_{22}' is no longer in as good an agreement with the exact values as for compressive loading. In the particular case $\frac{D_{23}'}{D_{22}'} = 0$, it is found that the exact results for the special-orthotropic plate can equally be applied to the plate oriented at 45° .

For the case of pure shear with D_{23}'/D_{22}' , the qualifying equations (75c) take the form

$$-2\alpha^2 - \beta^2 - \gamma^2 = 2 \frac{D_{33}'}{D_{22}'} \kappa^2$$

$$\alpha(\beta^2 - \gamma^2) = n_{xy}\kappa$$

$$(\alpha^2 - \beta^2)(\alpha^2 - \gamma^2) = \kappa^4$$

The equations are in agreement with equation (75d) of the special-orthotropic plate, when bearing in mind that, instead of the stiffness ratios D_{33}/D_{22} and D_{11}/D_{22} referred to the principal axes directions, the values D_{33}'/D_{22}' and D_{11}'/D_{22}' referred to $\omega = 45^\circ$ appear now, and that $\frac{D_{11}'}{D_{22}'} = 1$ for $\omega = 45^\circ$.

The exact buckling loads computed for the special-orthotropic plate by Seydel (reference 3) are also valid, provided $\frac{D_{23}'}{D_{22}'} = 0$, for the orthotropic plate with direction of fibers at 45° . The plate value $D_{33}/\sqrt{D_{11}D_{22}}$ again changes to D_{33}'/D_{22}' .

The unusual feature in figure 30 is the fact that the magnitude of the buckling loads in shear is essentially dependent upon the position of the principal axis of the greater bending stiffness with respect to the direction of the shear loading. Figure 31 represents two orthotropic plates of equal stiffness, one with the principal axis of maximum stiffness at 45° , the other at 135° relative to the plate edges.

At $\omega = 45^\circ$, $E_{11} > E_{22}$, as seen in figure 31, and $\frac{D_{23}'}{D_{22}'} < 0$ according to equation (24b); the buckling load - for equal stiffness ratio $\frac{D_{33}'}{D_{22}'}$ - assumes, according to figure 30, substantially higher values than for $\omega = 135^\circ$, where $E_{11} < E_{22}$ and hence $\frac{D_{23}'}{D_{22}'} > 0$.

In practical application, it means that by suitable orientation of the principal axis of maximum stiffness relative to the direction of the shear loads a substantial increase in the critical shear load of the plate can be obtained.

Figure 31 shows the ratio of the shear loads for these two cases plotted against the stiffness ratio D_{23}'/D_{22}' . The increase in buckling load of one over the other is readily seen. It assumes very considerable values with increasing $\frac{D_{23}'}{D_{22}'}$ - that is, with increasing difference of stiffness in the directions of the principal axes.

In figure 32 the approximated half-wave lengths of the buckling surface formed under the critical load are represented in comparison with the exact values for several stiffness combinations of table 3. The agreement is very good.

The approximate value ϵ of the slope of the nodal lines of the buckling surface can be taken from figure 33.

3. Buckling Loads of Plywood

The stiffness values for $\omega = 45^\circ$ were computed on several plywood plates of different construction with the moduli of elasticity indicated in figure 14, and the buckling stresses, half-wave lengths and nodal line inclinations determined for compression and shear loading with the aid of figures 25 to 28 and 29 to 33. The results were then plotted against the plywood construction in figures 34 and 35.

4. Example

The application of the results described in sections 1 and 2 is illustrated on a worked-out model problem:

Consider a very long, freely supported plywood plate built up of three identically thick plies, the grains of which are oriented at $\omega = 45^\circ$ and 135° relative to the plate edge. The plate is $b = 2a = 10$ cm in height, by $s = 1$ mm in wall thickness.

Young's modulus E for the principal axes of the plate can be determined by experiment or taken from figure 14.

According to figure 14, they are:

$$E_{11} = 176,500 \frac{\text{kg}}{\text{cm}^2}$$

$$E_{22} = 11,500 \frac{\text{kg}}{\text{cm}^2}$$

$$\nu_{11} = 0.196$$

$$\nu_{22} = 0.0130$$

$$G = 10,500 \frac{\text{kg}}{\text{cm}^2}$$

With these data, the stiffness values for the axes at 45° can be computed by (29a) and (33)

$$D_{11}' = D_{22}' = \frac{s^3}{12} \left(G + \frac{1}{4} \frac{E_{11} + E_{22} + 2\nu_{11}E_{22}}{1 - \nu_{11}\nu_{22}} \right)$$

$$= \frac{s^3}{12} (10,500 + 47,300) = \frac{s^3}{12} 57,800 = 4.81 \text{ kgcm}$$

$$D_{33}' = \frac{s^3}{12} \left(-G + \frac{1}{4} \frac{E_{11} + E_{22} + 2\nu_{11}E_{22}}{1 - \nu_{11}\nu_{22}} + \frac{1}{2} \frac{E_{11} + E_{22} - 2\nu_{11}E_{22}}{1 - \nu_{11}\nu_{22}} \right)$$

$$= \frac{s^3}{12} (-10,500 + 47,300 + 91,000) = \frac{s^3}{12} 127,800 = 10.64 \text{ kgcm}$$

$$D_{13}' = D_{23}' = \frac{s^3}{12} \left(\frac{1}{4} \frac{E_{22} - E_{11}}{1 - \nu_{11}\nu_{22}} \right) = -\frac{s^3}{12} 40,550 = -3.38 \text{ kgcm for } \omega = 45^\circ$$

and

$$= \frac{s^3}{12} 40,550 = 3.38 \text{ kgcm for } \omega = 135^\circ$$

With these values the stiffness ratios become

$$\frac{D_{23}'}{D_{22}'} = \pm \frac{3.38}{4.81} = \pm 0.701$$

$$\frac{D_{33}'}{D_{22}'} = \frac{10.64}{4.81} = 2.20$$

The buckling load of the plate stressed in compression for this pair of values follows then from figure 25 as

$$n_x = \frac{N_x a^2}{D_{22}} = 6.89$$

and hence the buckling stress as

$$\sigma_x = \frac{N_x}{s} = \frac{n_x D_{22}}{a^2 s} = \frac{6.89 \times 4.81}{25 \times 0.1} = 13.3 \text{ kg/cm}^2$$

The half-wave length of the wrinkles follows from figure 27:

$$L = \frac{L}{b} b = 0.817 \times 10 = 8.17 \text{ cm}$$

For the plate stressed in shear, figures 29 and 32 show

(1) for $\omega = 45^\circ$

$$\tau_{kr} = \frac{n_{xy}}{s} = \frac{n_{xy} D_{22}}{a^2 s} = \frac{4.56 \times 4.81}{25 \times 0.1} = 8.8 \text{ kg/cm}^2$$

and

$$L = \frac{L}{b} b = 1.76 \times 10 = 17.6 \text{ cm}$$

(2) for $\omega = 135^\circ$

$$\tau_{kr} = \frac{n_{xy} D_{22}}{a^2 s} = \frac{31.65 \times 4.81}{25 \times 0.1} = 61.0 \text{ kg/cm}^2$$

and

$$L = \frac{L}{b} b = 0.89 \times 10 = 8.9 \text{ cm}$$

D. BUCKLING LOADS OF THE GENERAL ORTHOTROPIC PLATE REFERRED TO ω
(PLYWOOD OF VARYING CONSTRUCTION)

In view of the large amount of paper work involved in a general representation of the buckling loads of the general orthotropic plate relative to the stiffness factors and angle ω , due to the great number of parameters, the characteristic relationship between the buckling loads and the angle ω is exemplified on a worked out model problem.

The results were obtained by approximation method, the known exact values for $\omega = 0^\circ$, 45° , and 90° are also given for comparison. The results are again limited to a very long plate strip freely supported at the edges.

The calculation is based on the modulus of elasticity of plywood stressed in bending, indicated in table 1.

1. Results for Pure Compression

In figure 36 the buckling stresses in pure compression are represented for $\omega = 0^\circ$ to 180° . Owing to the symmetry of the compressive loading, the buckling stresses are symmetrical to the value $\omega = 90^\circ$. They are of equal magnitude for plywood with and across the grain ($\omega = 0^\circ$ and 90°) and increase for intermediate angles ω . The maximum buckling stress for plywood of very many laminations (∞) is reached at $\omega = 45^\circ$ and 135° , respectively.

The half-wave lengths of the plate represented in figure 37 indicate that, if the plate consists of a few plies, the buckling length, with the grain, is greater than across the grain. The buckling load increased by the greater longitudinal stiffness at $\omega = 0^\circ$ relative to $\omega = 90^\circ$, is reduced again by the greater buckling length so that the buckling loads are the same again in spite of the substantially different longitudinal stiffnesses for $\omega = 0^\circ$ and $\omega = 90^\circ$.

The ensuing average nodal lines which, as under compressive loading, are inclined to the plate edges according to section IIIc, and at right angles to the edges, in a few specific cases only, are shown in figure 38.

2. Results for Pure Shear Loading

On account of the unsymmetry of the shear loading the buckling stresses are no longer symmetrical to $\omega = 90^\circ$.

The close relationship existing between shearing stresses and angle ω - especially for plywood with few laminations - is seen from figure 39. Owing to the unsymmetry of the plywood structure as well as to the shear loading, the maximum shearing stress occurs at about 60° rather than for $\omega = 45^\circ$.

The half-wave lengths and the slope of the nodal lines in figures 40 and 41 also exhibit a marked dependence on angle ω , the half-wave length - for three-plywood - fluctuates between $\frac{L}{b} = 2.64$ and 0.56 .

For an estimate of the error introduced by the approximate as against the exact solution, the exact values for $\omega = 0^\circ$ and 90° are represented in figures 39 and 40 according to the results by Seydel and for $\omega = 45^\circ$ according to figure 29. For $\omega = 45^\circ$ the maximum error is 12 percent; for other cases the error is less. The agreement between the approximated and the exact half-wave lengths is remarkably good. The approximate calculation which involves incomparably less paper work than the exact solution yields a very good insight into the relationship between the buckling stresses and buckling lengths and angle ω , notwithstanding the cited differences.

Translated by J. Vanier
National Advisory Committee
for Aeronautics

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TABLE 1.— EFFECT OF PLATE CONSTRUCTION OF PLANE AND FLEXURALLY
STRESSED PLYWOOD PLATES ON THE MODULI OF ELASTICITY

Number of plies	Plate Stressed in plane									Plate stressed in bending								
	D	$\omega = 0^\circ$					$\omega = 45^\circ$			D	$\omega = 0^\circ$					$\omega = 45^\circ$		
		E_{11}	E_{22}	ν_{11}	ν_{22}	G	$E_{11}' = E_{22}'$	$\nu_{11}' = \nu_{22}'$	G'		E_{11}	E_{22}	ν_{11}	ν_{22}	G	$E_{11}' = E_{22}'$	$\nu_{11}' = \nu_{22}'$	G'
1	0	180,000	5,000	0.45	0.0125	10,500	11,700	-0.113	4,750	0	180,000	5,000	0.45	0.0125	10,500	11,700	-0.113	4,750
3	0.5	121,600	63,400	0.0355	0.0185	10,500	33,780	0.604	40,670	0.0385	173,500	11,500	0.1959	0.0130	10,500	21,570	0.035	10,520
5	0.667	110,000	75,000	0.0301	0.0205	10,500	34,210	0.626	43,520	0.263	143,600	41,400	0.0544	0.0157	10,500	31,900	0.516	31,370
7	0.75	105,000	80,000	0.0281	0.0214	10,500	34,330	0.631	44,340	0.406	129,500	55,500	0.0405	0.0174	10,500	33,300	0.576	37,940
11	0.833	100,400	84,600	0.0266	0.0224	10,500	34,400	0.635	44,820	0.573	116,200	68,800	0.0327	0.0194	10,500	34,050	0.616	42,180
∞	1.00	92,500	92,500	0.0243	0.0243	10,500	34,460	0.637	45,140	1.000	92,500	92,500	0.0243	0.0243	10,500	34,460	0.637	45,140

TABLE 2.—BUCKLING LOADS OF THE SPECIAL-ORTHOTROPIC PLATE STRIP,
IN PURE COMPRESSION VERSUS CHARACTERISTIC VALUE OF THE PLATE

Characteristic value δ	Supported plate	Clamped plate approximateion according to		
	Exact values	Exact values	Equation (87)	Equation (87a)
0	$\frac{\pi^2}{2} = 4.935$	$\frac{\pi^2}{2} = 4.935$	6.60	5.80
0.1	5.43	6.69	7.74	6.94
0.2	5.92	7.98	8.88	8.08
0.3	6.42	9.15	10.02	9.22
0.4	6.91	10.35	11.16	10.36
0.6	7.90	12.64	13.44	12.64
0.707	8.42	13.88	14.58	13.88
0.8	8.88	14.92	15.72	14.92
0.9	9.38	16.06	16.86	16.06
1.0	$\pi^2 = 9.87$	17.20	18.00	17.20
1.414	11.91	21.96	22.72	21.92
1.582	12.74	23.95	24.61	23.82
2.0	14.81	28.52	29.40	28.60
2.5	17.27	34.00	35.10	34.30
3.33	21.38	43.22	44.58	43.80
5.0	29.61	61.75	63.60	62.80

TABLE 3.- COMPARISON OF THE RESULTS FOR PURE SHEAR ($\omega = 45^\circ$) ACCORDING TO THE APPROXIMATE METHOD WITH THE ACTUAL RESULTS

	$\frac{D_{23}'}{D_{22}'} = 1.0$				$\frac{D_{23}'}{D_{22}'} = 0.5$				$\frac{D_{23}'}{D_{22}'} = 0$				$\frac{D_{23}'}{D_{22}'} = -0.5$				$\frac{D_{23}'}{D_{22}'} = -1.0$			
$\frac{D_{33}'}{D_{22}'}$	Exact		Approximate		Exact		Approximate		¹ Exact		Approximate		Exact		Approximate		Exact		Approximate	
	n_{xy}	$\frac{L}{b}$	n_{xy}	$\frac{L}{b}$	n_{xy}	$\frac{L}{b}$	n_{xy}	$\frac{L}{b}$	n_{xy}	$\frac{L}{b}$	n_{xy}	$\frac{L}{b}$	n_{xy}	$\frac{L}{b}$	n_{xy}	$\frac{L}{b}$	n_{xy}	$\frac{L}{b}$	n_{xy}	$\frac{L}{b}$
0	---	---	---	---	-----	-----	-----	-----	8.125	1.025	8.77	1.021	-----	-----	-----	-----	-----	-----	-----	-----
1	---	---	---	---	2.32	-----	2.37	0.43	13.165	1.245	13.98	1.225	20.96	1.45	23.40	1.45	-----	-----	-----	-----
2	---	---	---	---	8.07	1.126	8.22	1.105	17.25	1.462	18.22	1.455	24.87	1.69	27.40	1.66	-----	-----	-----	-----
3	0	0	0	0	12.06	1.45	12.35	1.414	20.50	1.715	21.80	1.682	28.19	1.911	30.78	1.86	34.56	1.98	39.50	2.00

¹According to Seydel (reference 3), figure 4a and 6a.

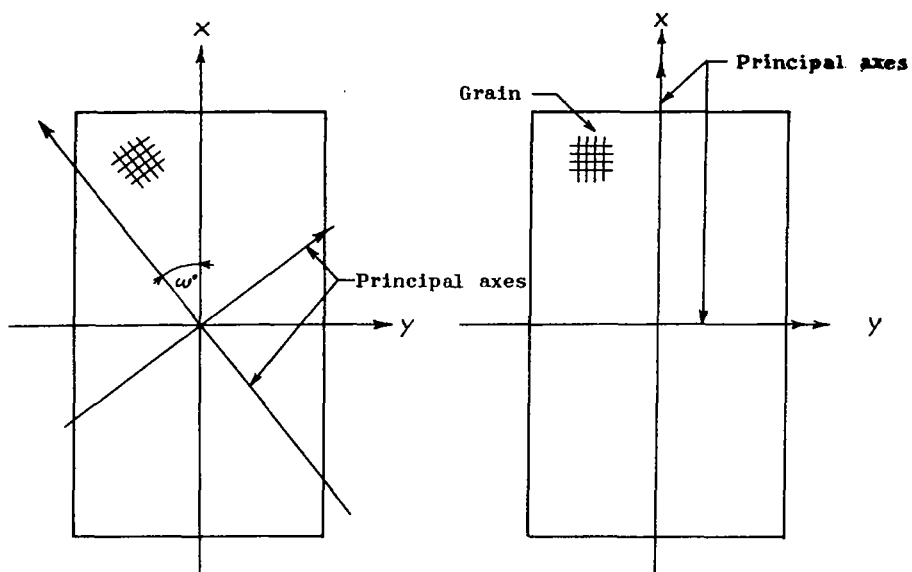


Figure 1(a).- General-orthotropic plate. Figure 1(b).- Special-orthotropic plate.

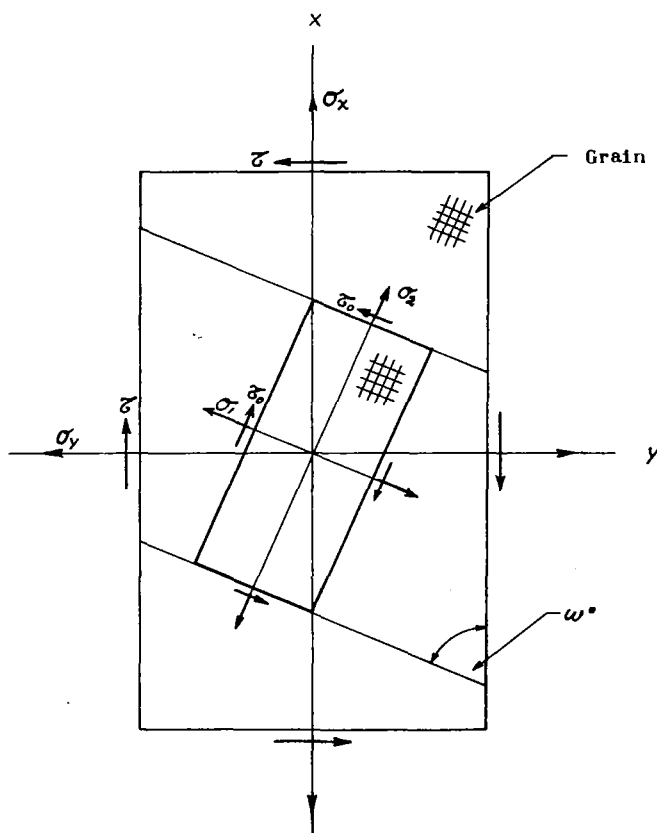
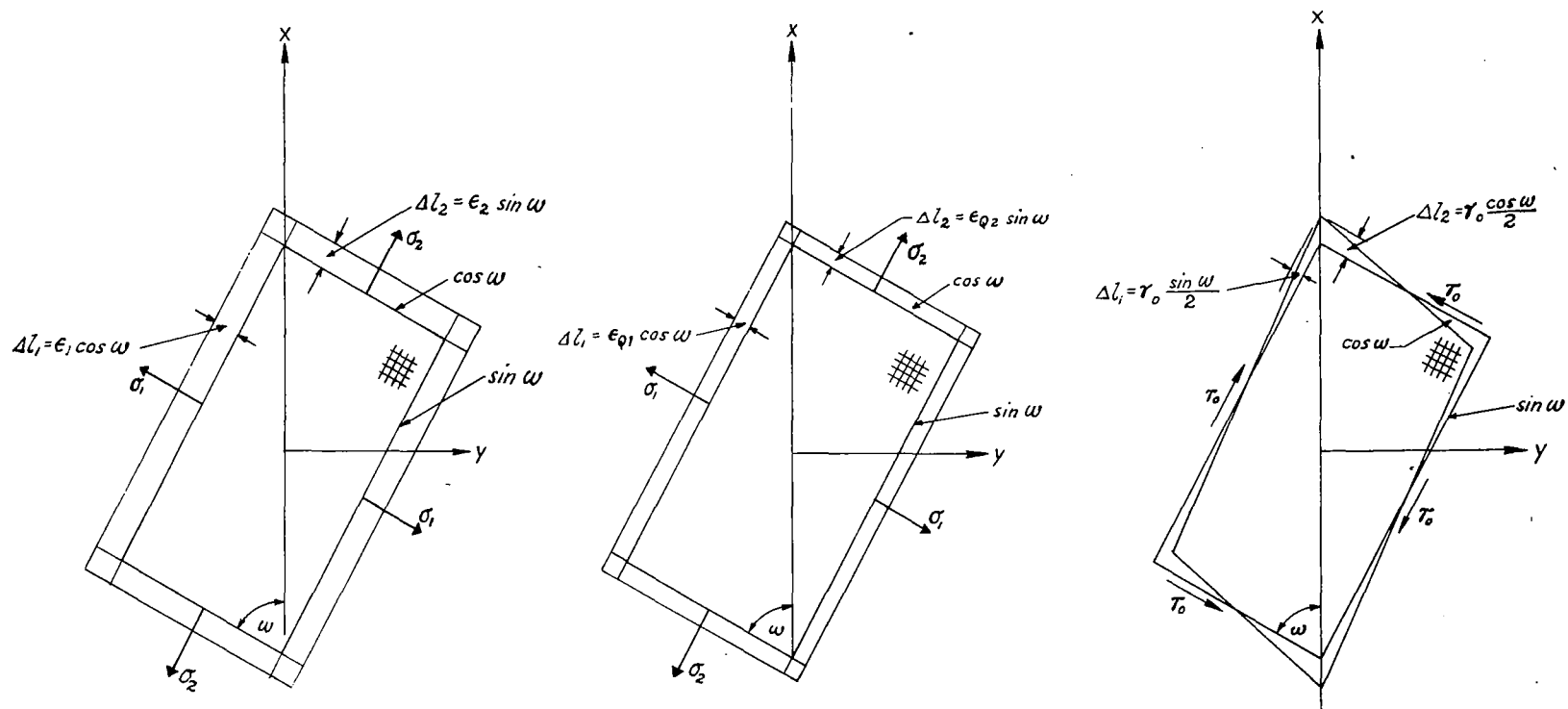


Figure 2.- Stresses on an orthotropic plate element.



Figures 4(a)-(c).- Strain components Δl_1 and Δl_2 on the plate element due to longitudinal deformation, transverse contraction, and shearing deformation.

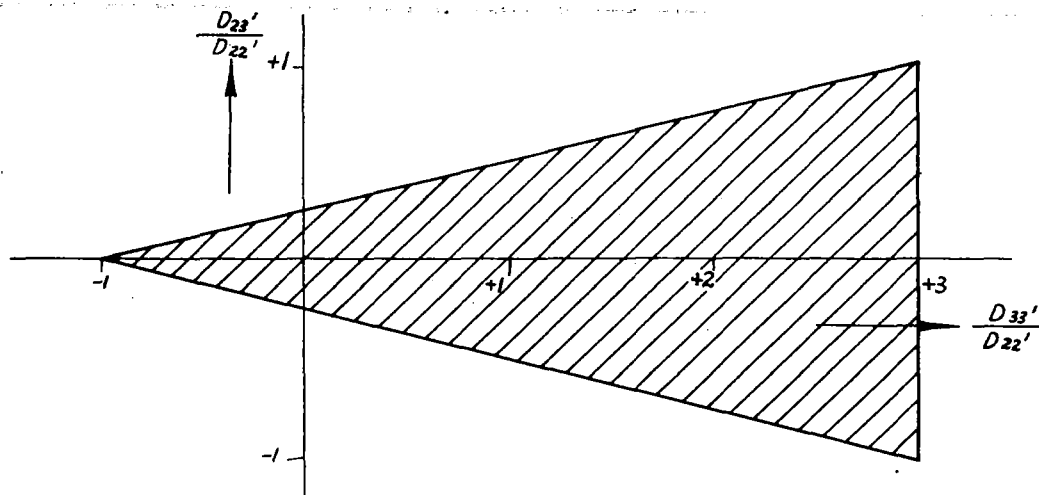


Figure 5(a).- Range of existence of factors $\frac{D_{23}'}{D_{22}'}$, $\frac{D_{33}'}{D_{22}'}$.

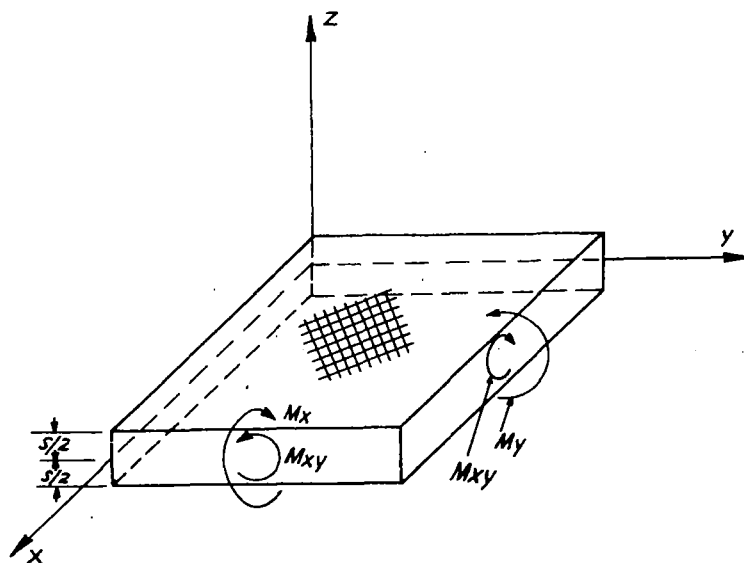


Figure 5(b).- Section moment of a plate element.

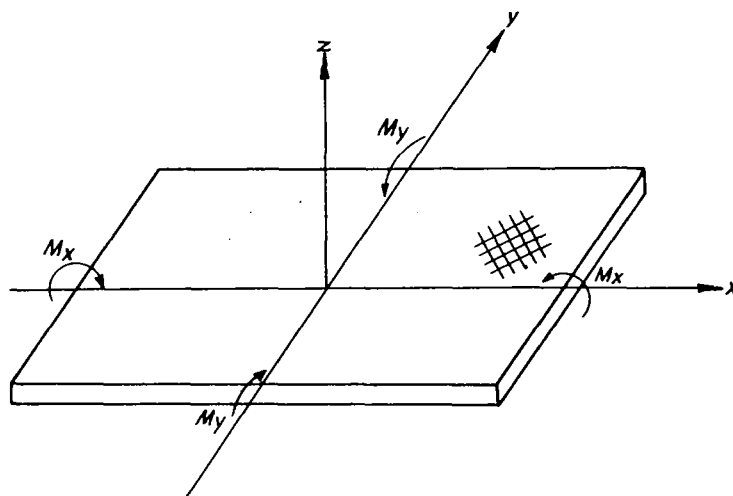


Figure 6.- Pure axial stress condition in a plate.

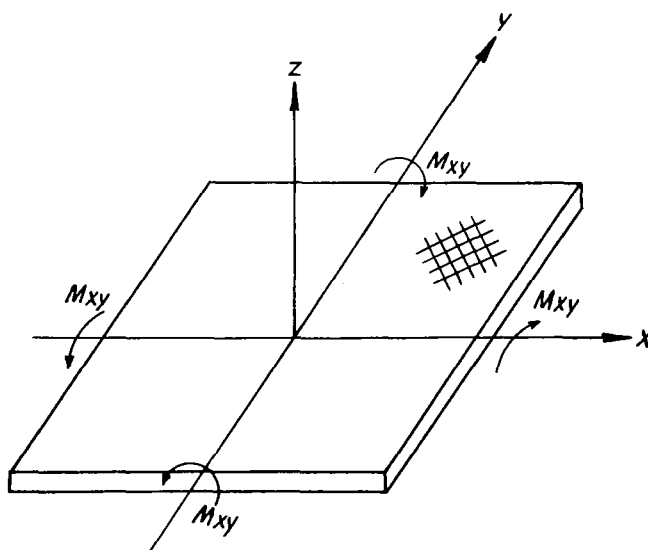


Figure 7.- Pure shearing stress condition in a plate.

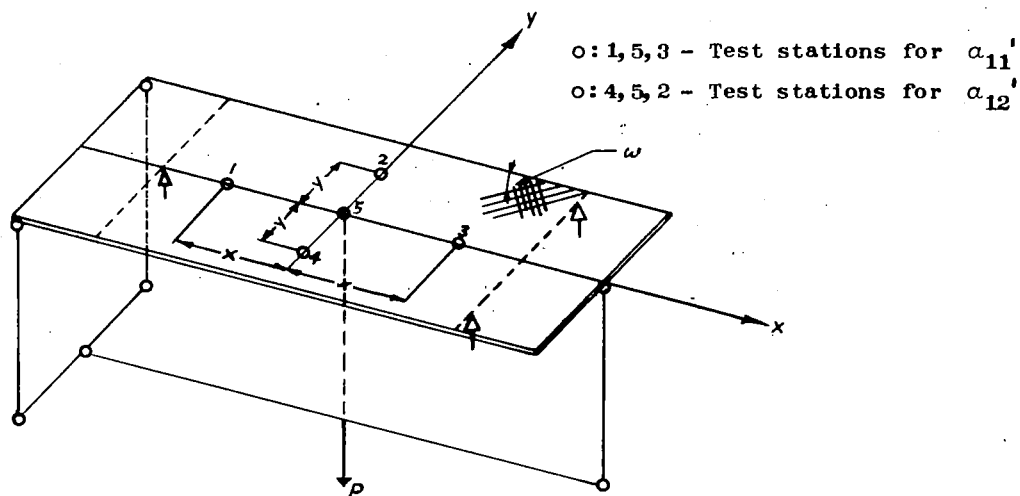


Figure 8.- Experimental setup for defining modulus of elasticity α_{11}' , α_{22}' , and α_{12}' .

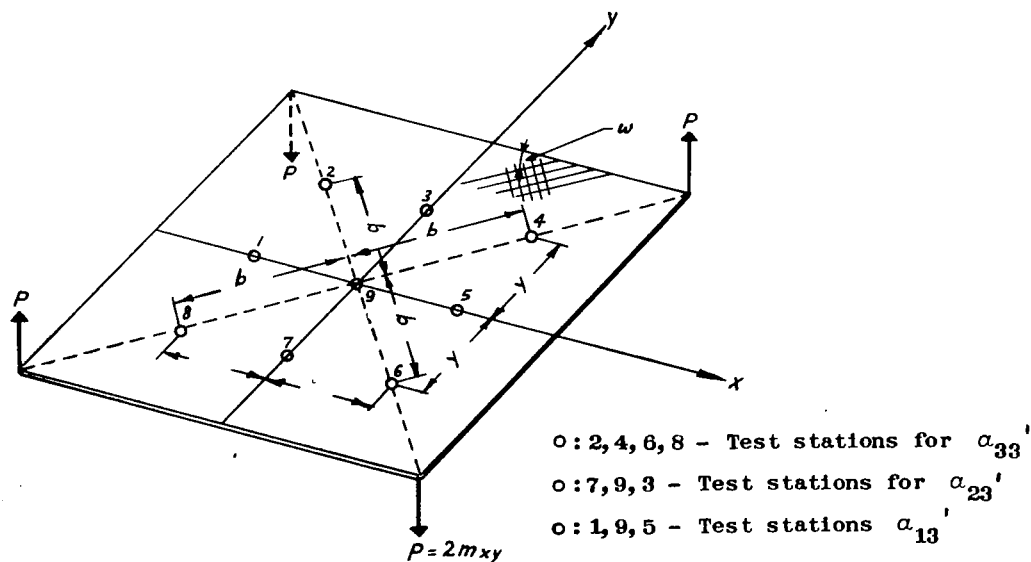


Figure 9.- Experimental setup for defining modulus of elasticity α_{13}' , α_{23}' , and α_{33}' .

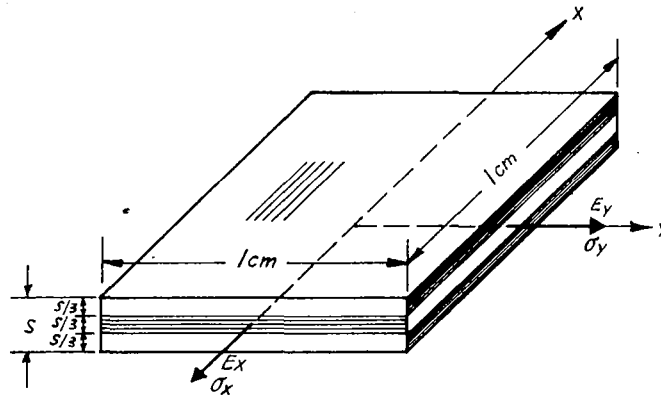


Figure 10.- Construction of a plywood element.

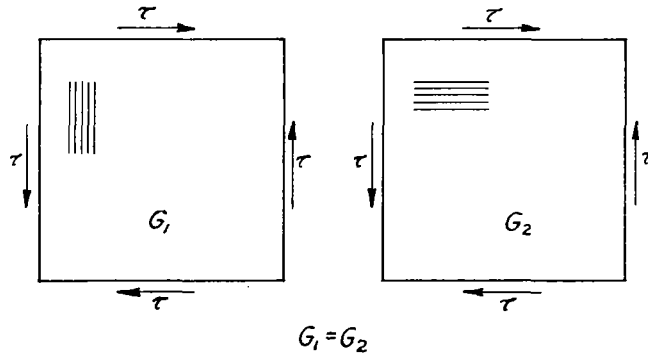


Figure 11.- For determining the modulus of rigidity of plywood.

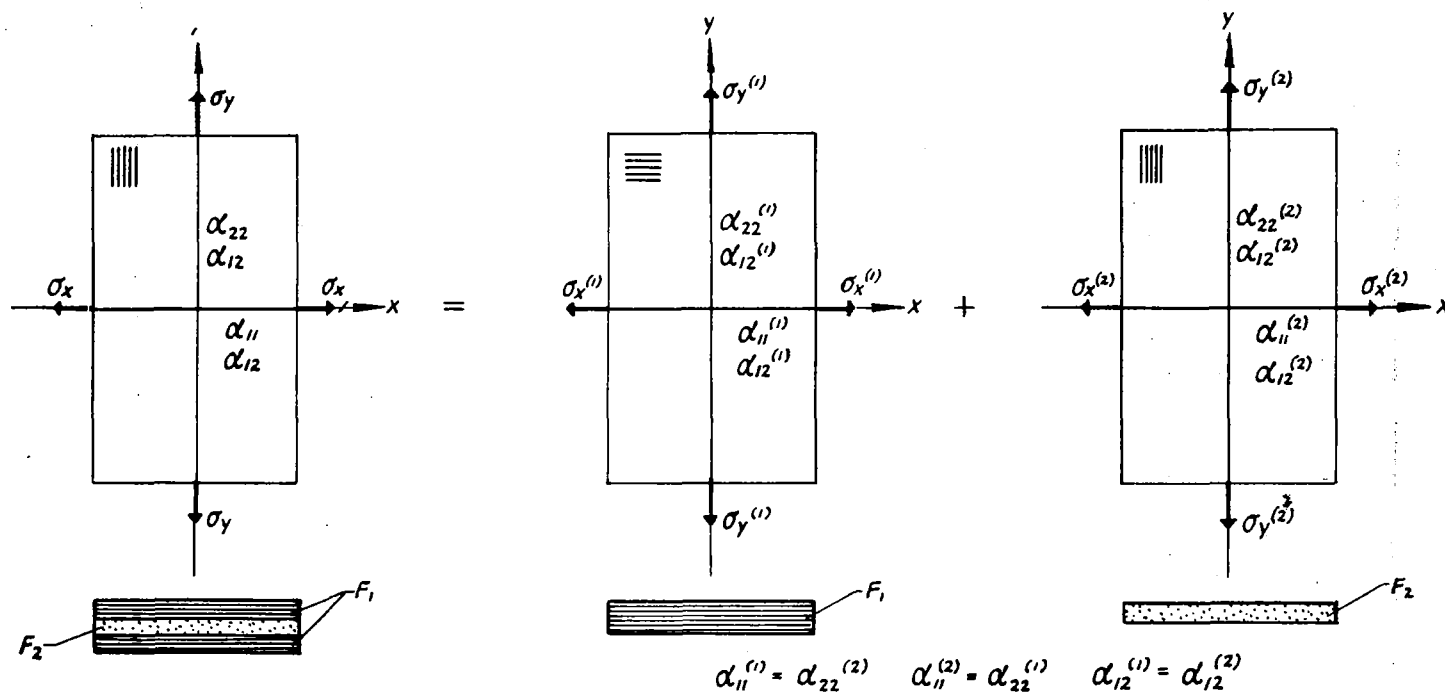


Figure 12.- Division of plywood into its proportional laminations.

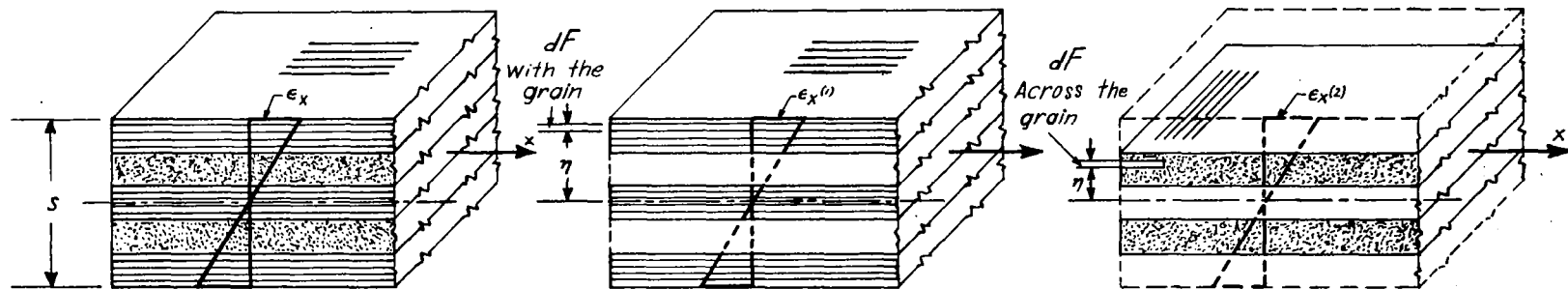


Figure 13.- Assumed flexural strain distribution of plywood.

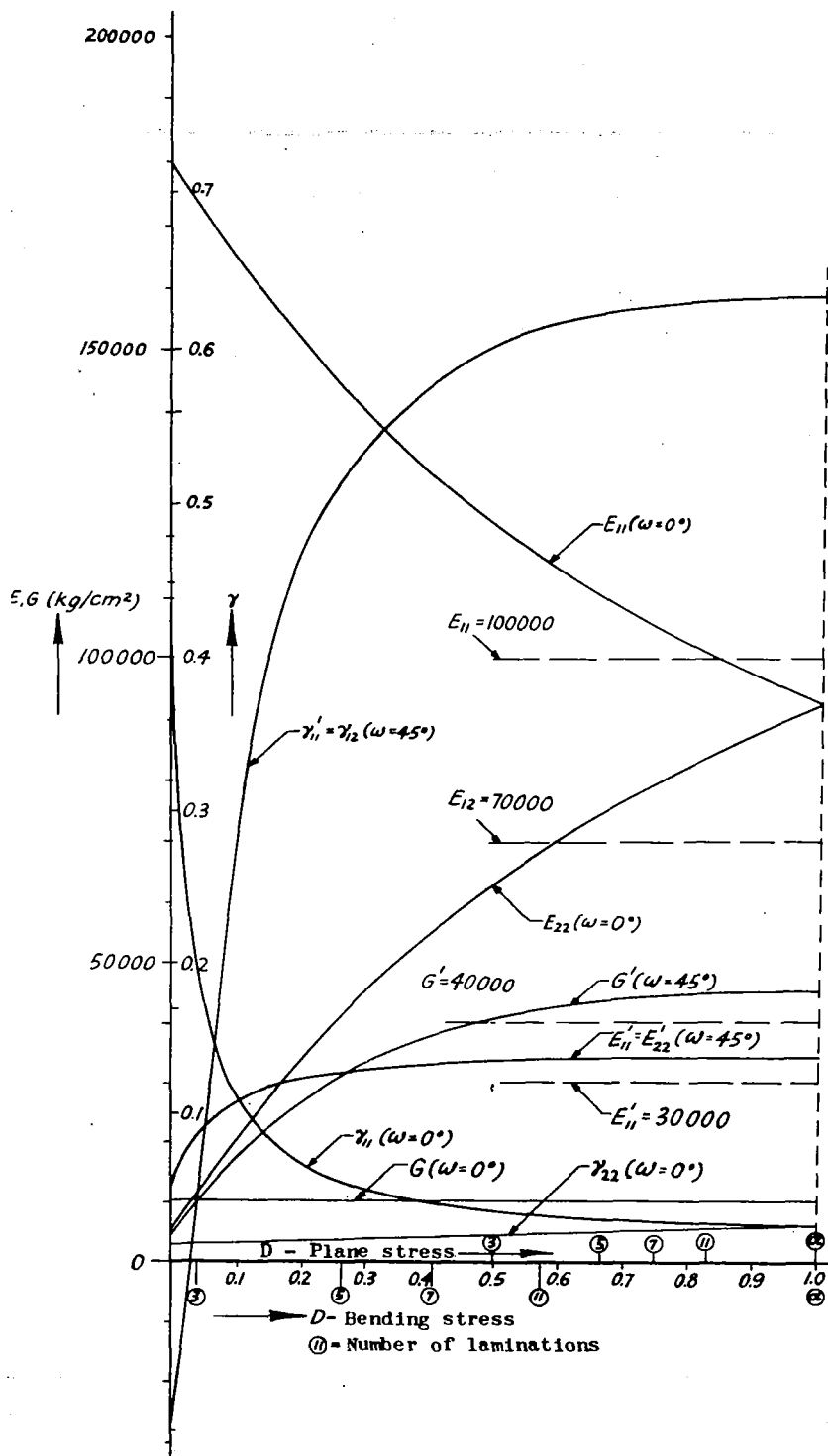


Figure 14.- Variation of modulus of elasticity of a plywood plate under loads at 0° and 45° plotted against construction of plate.

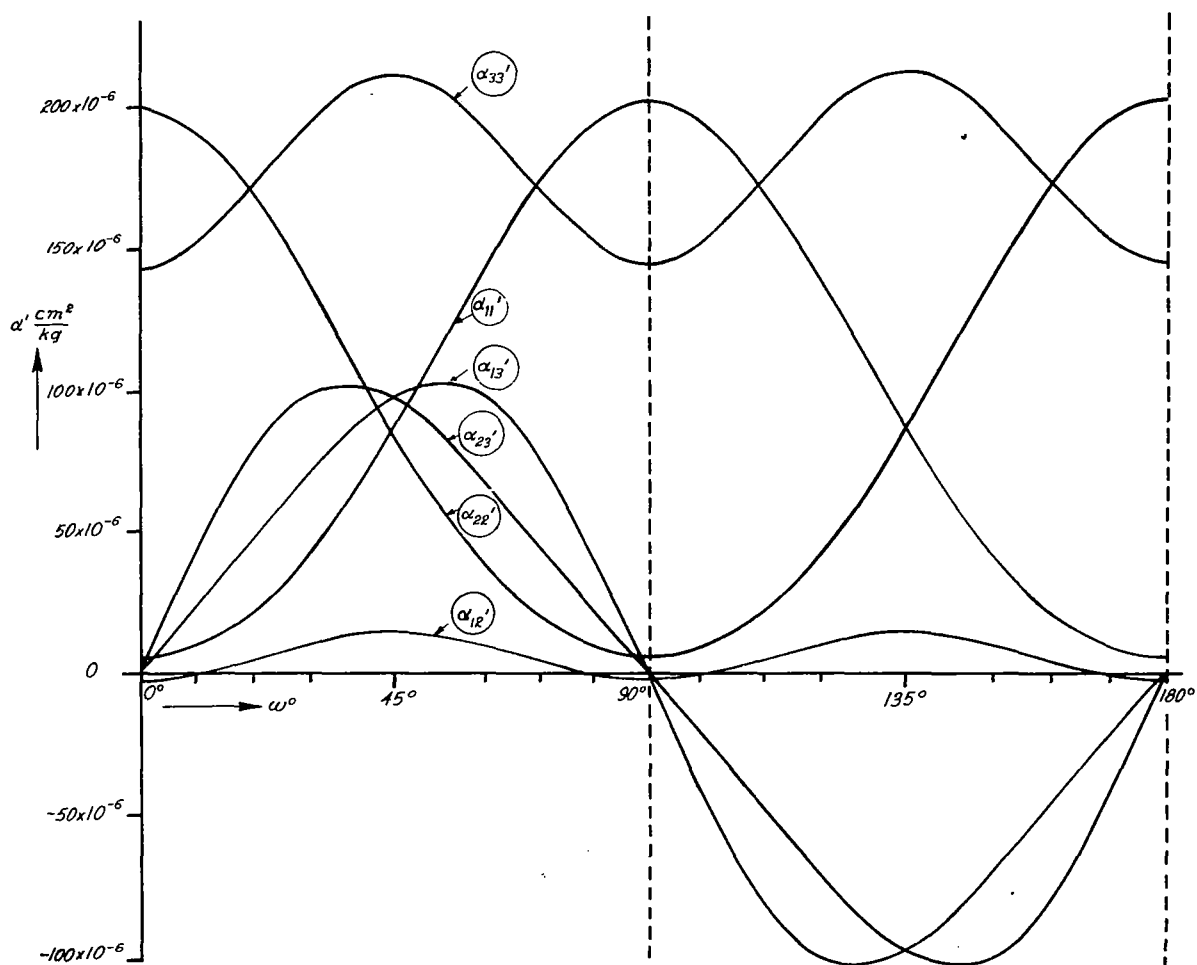


Figure 15.- Relationship between modulus of elasticity α' of a veneer and ω .

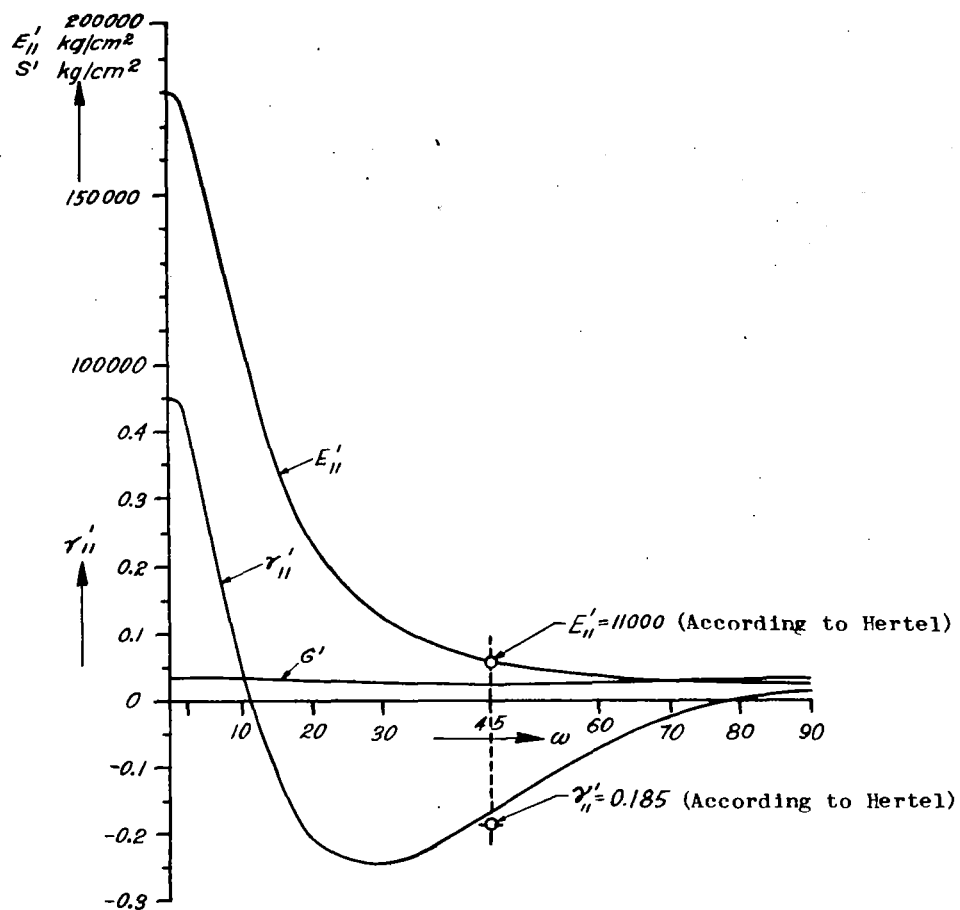


Figure 16.- Relationship between modulus of elasticity E'_{11} , γ'_{11} , and G' and ω .

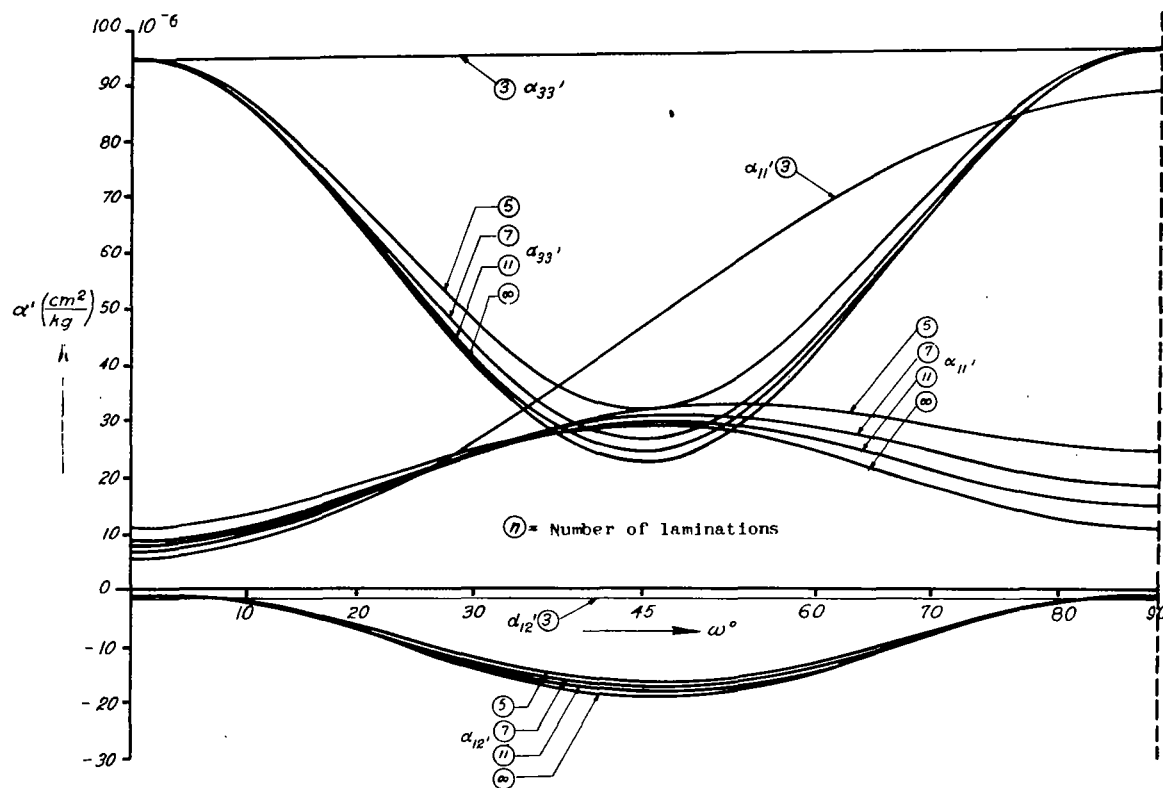


Figure 17.- Relationship between modulus of elasticity α' of plane stressed plywood plates and ω .

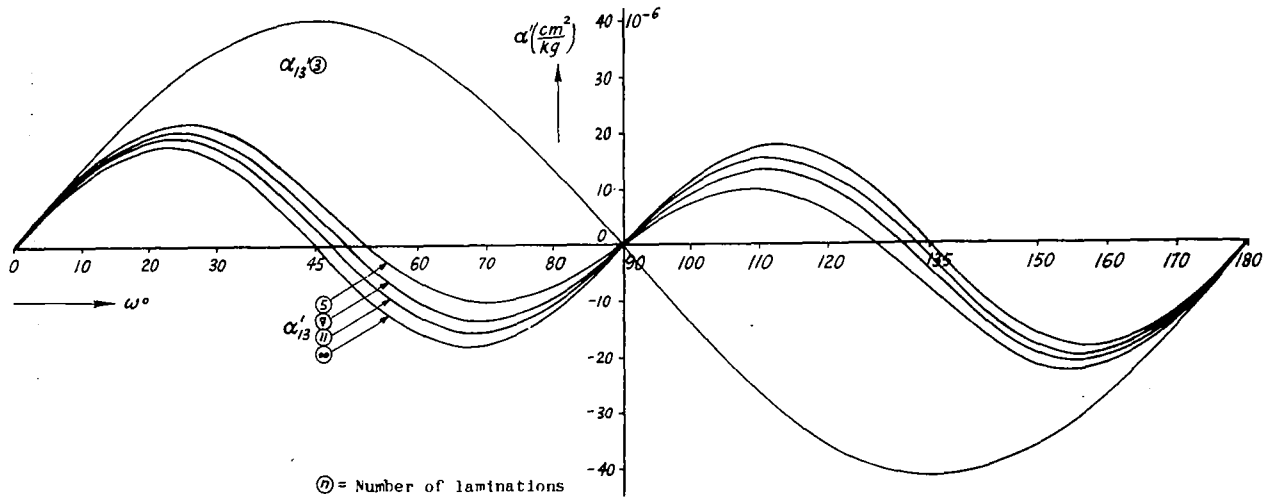


Figure 18(a).

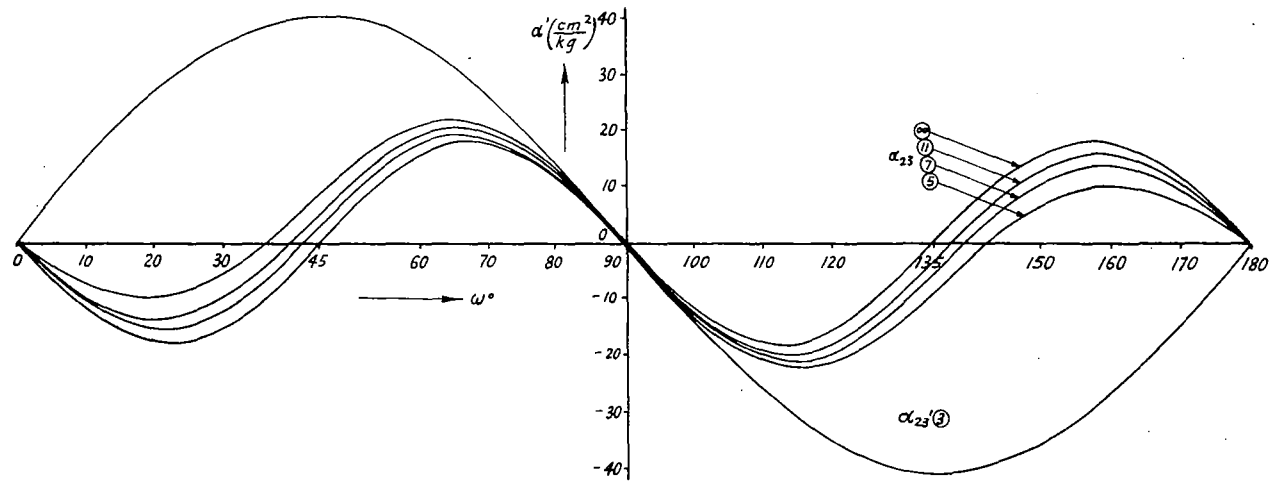


Figure 18(b).

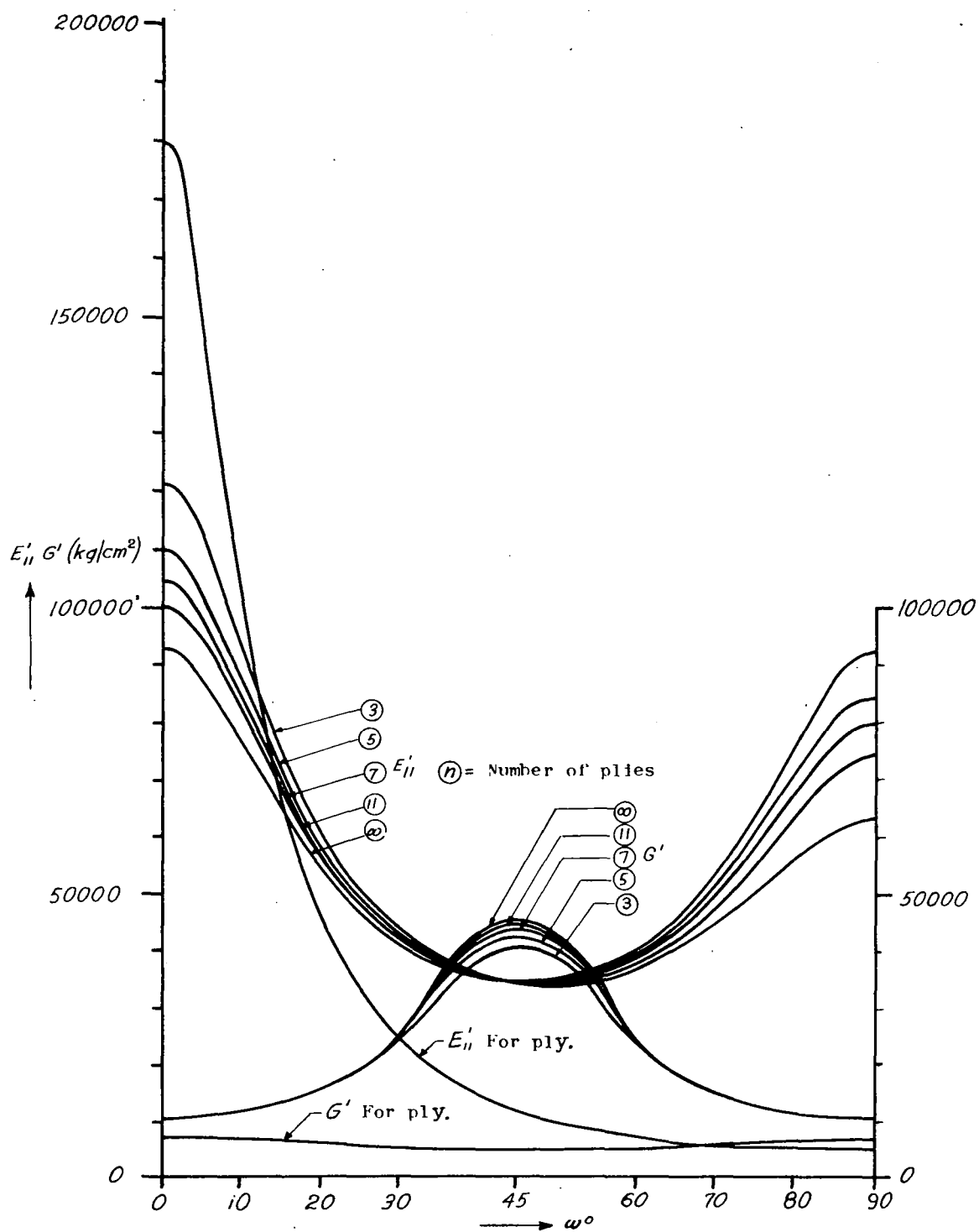


Figure 19.- Relationship between modulus E'_{11} and G' of plane stressed plywood plates and ω .

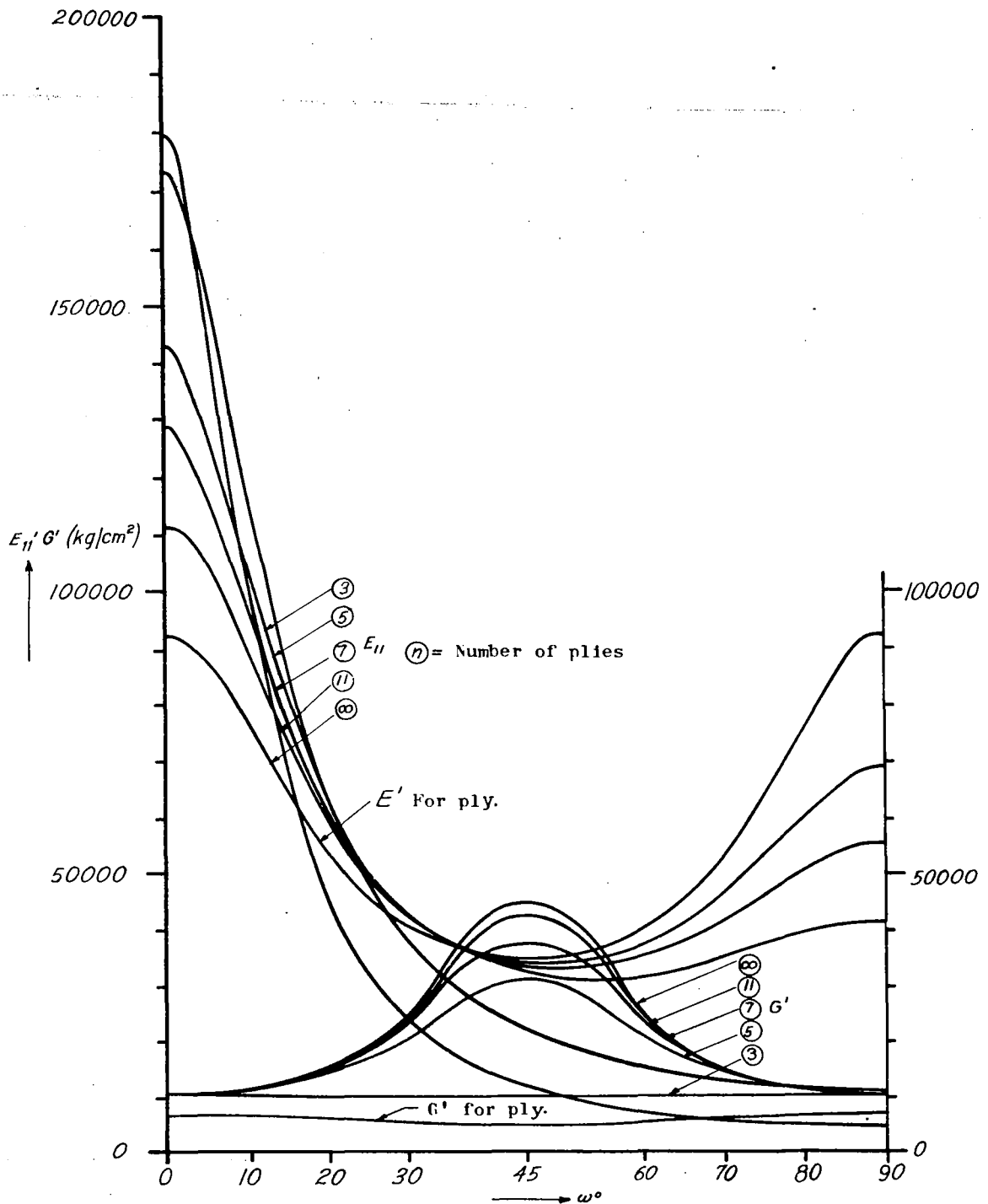


Figure 20.- Relationship between modulus E_{11}' and G' of flexurally stressed plywood plates and ω .

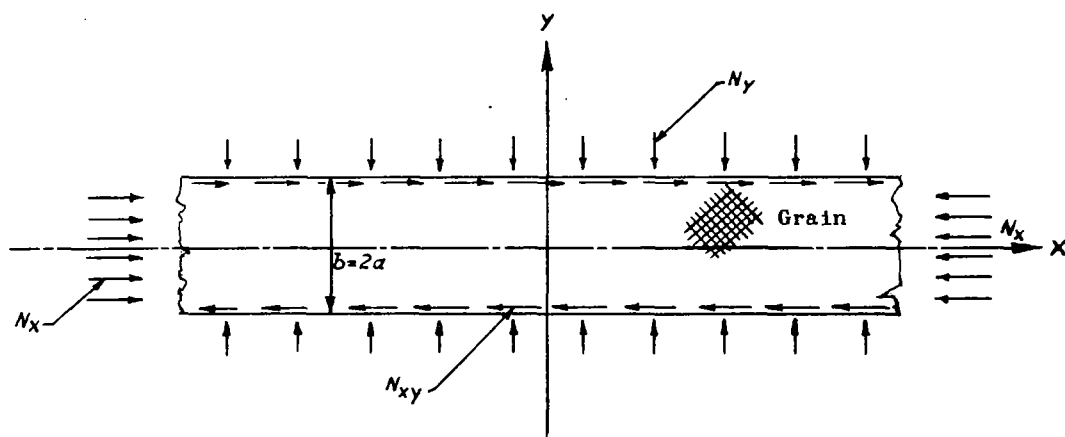


Figure 21.- Orthotropic plate strip loaded in tension and shear.

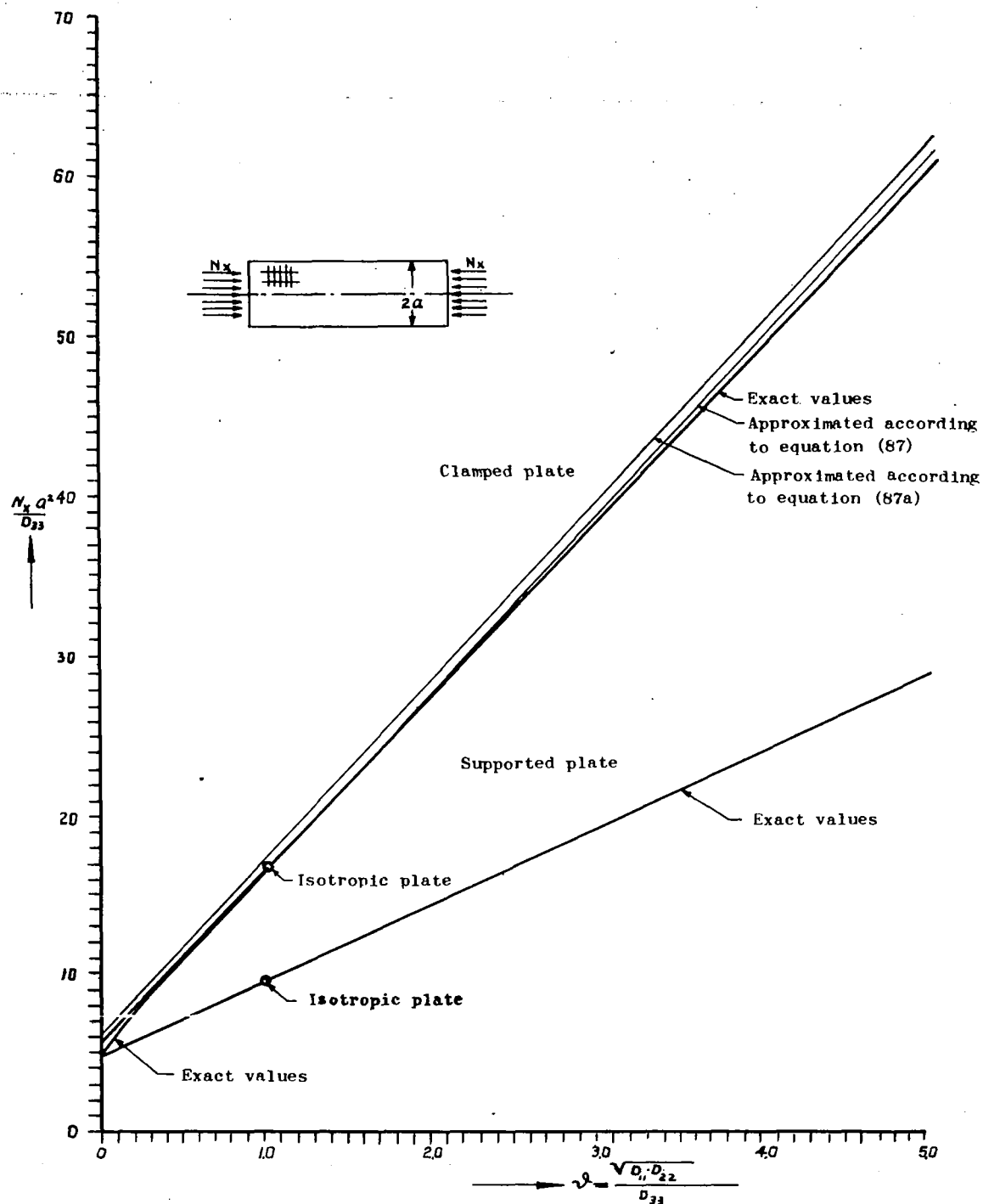


Figure 22.- Buckling loads of a special-orthotropic plate under axial stress.

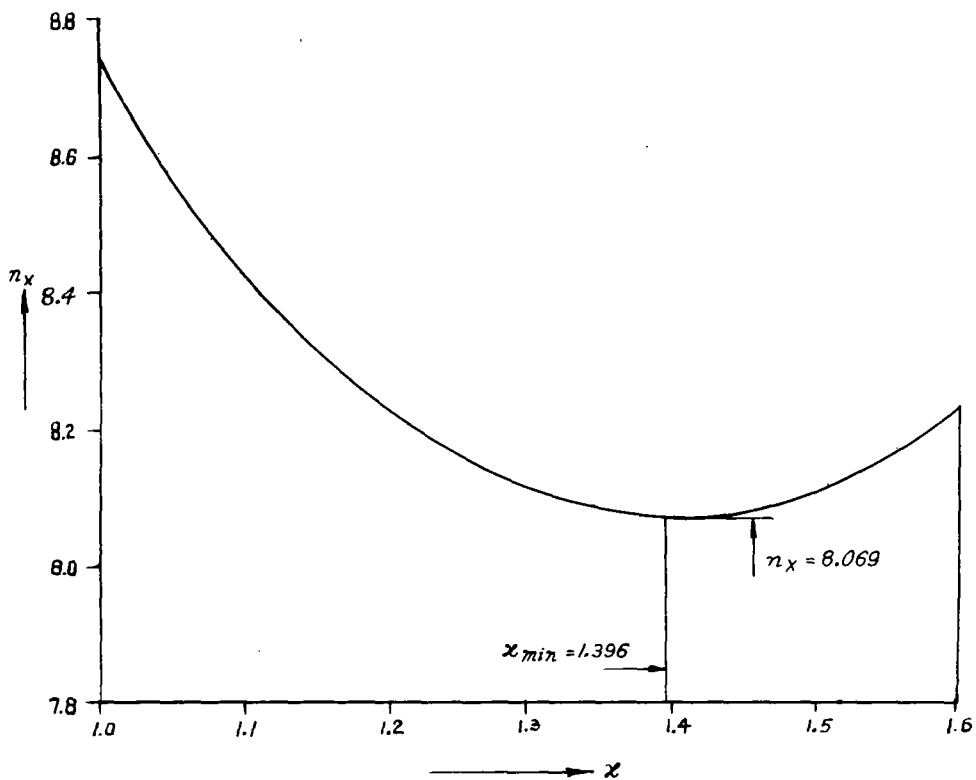


Figure 23.- Buckling load of a plate strip referred to κ
 (for the example $\frac{D_{23}}{D_{22}} = -0.5, \frac{D_{33}}{D_{32}} = 2.0$)

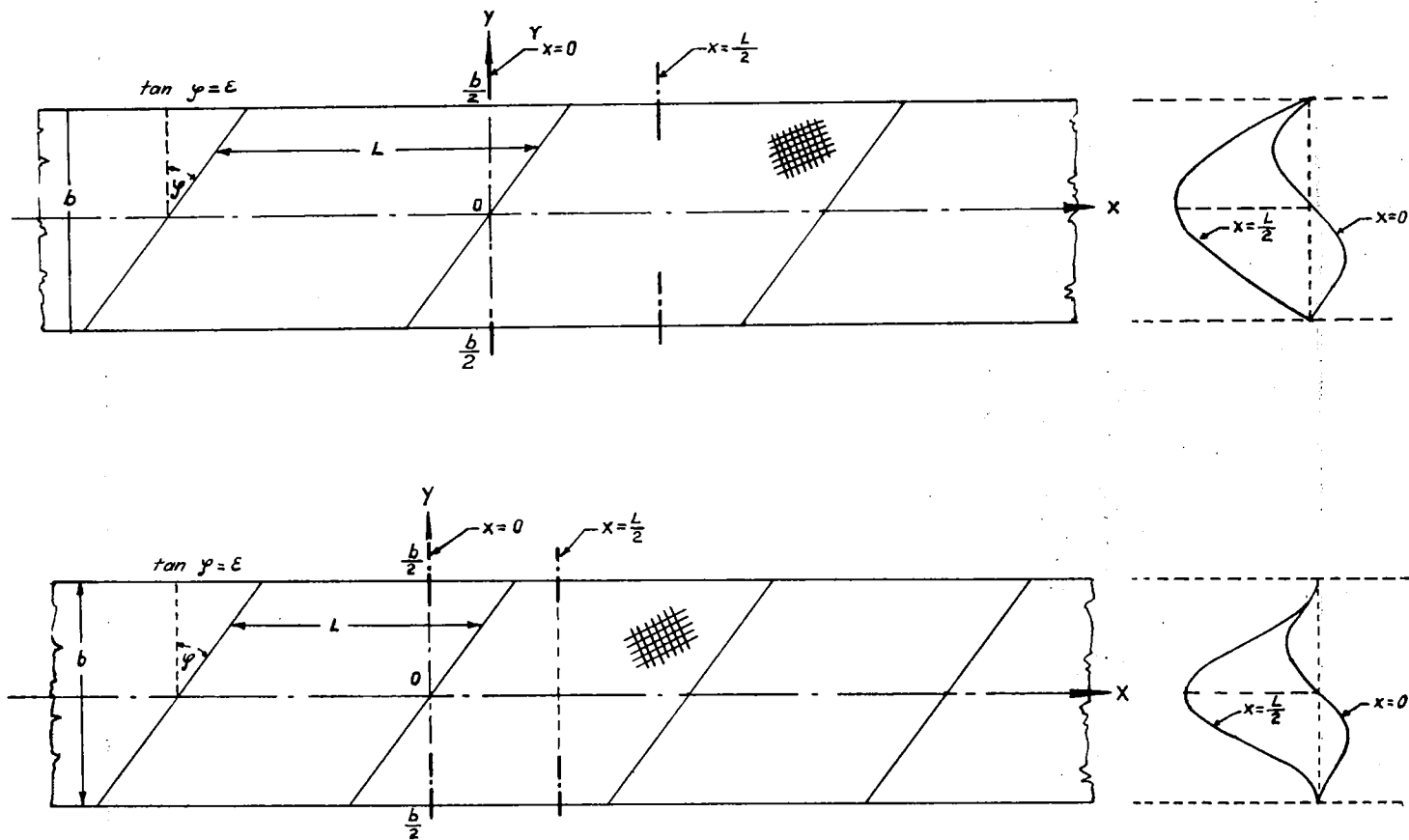


Figure 24.- Approximate assumptions for the buckling surface of the general-orthotropic plate for freely supported and clamped plate edges.

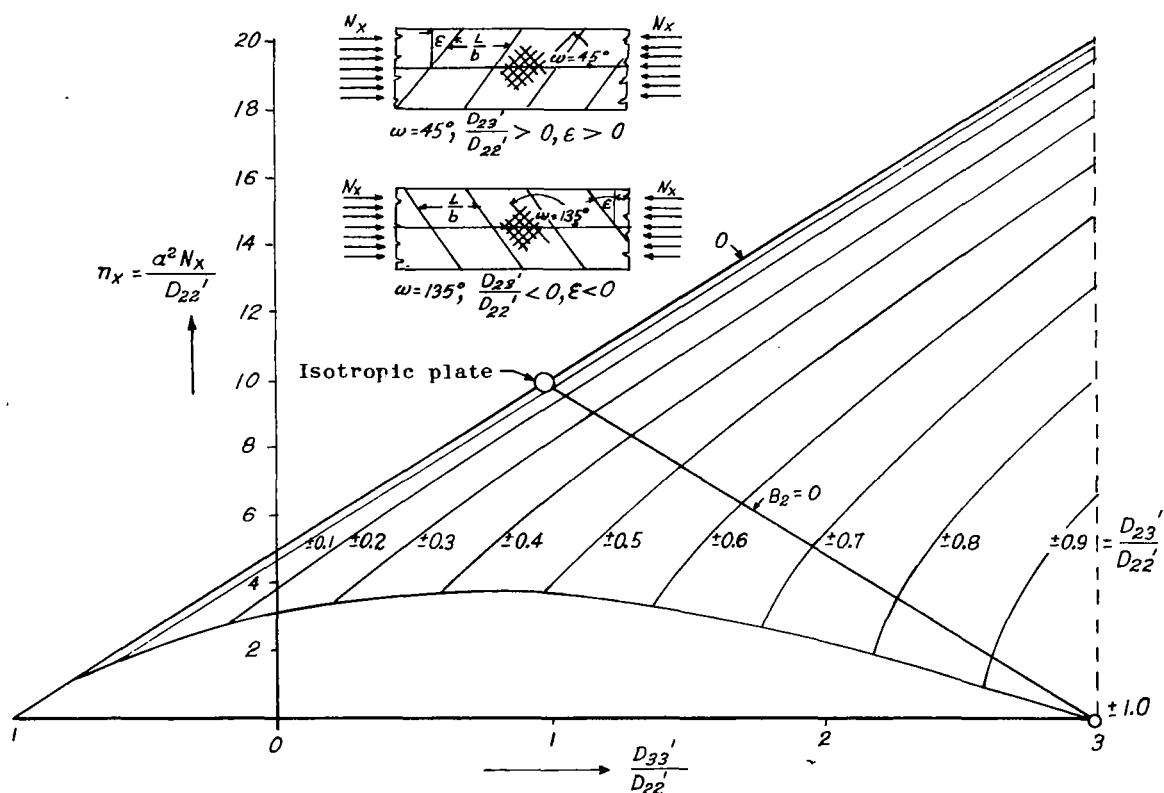


Figure 25.- Buckling load of orthotropic plate strip under axial forces ($\omega = 45^\circ$ and 135°) (approximate values for free support).

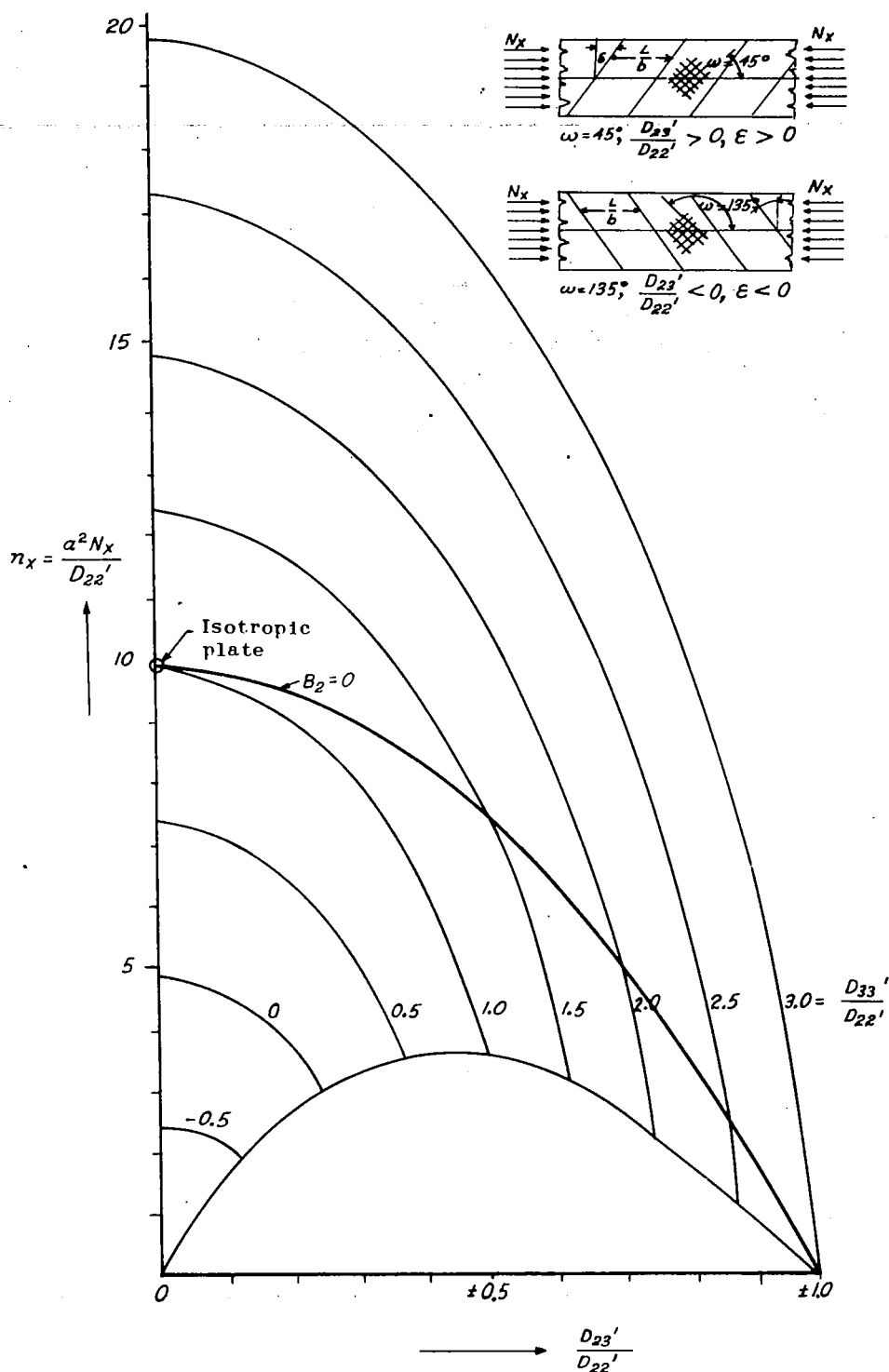


Figure 26.- Buckling load of an orthotropic plate strip under axial forces ($\omega = 45^\circ$ and 135°) (approximate values for free support).

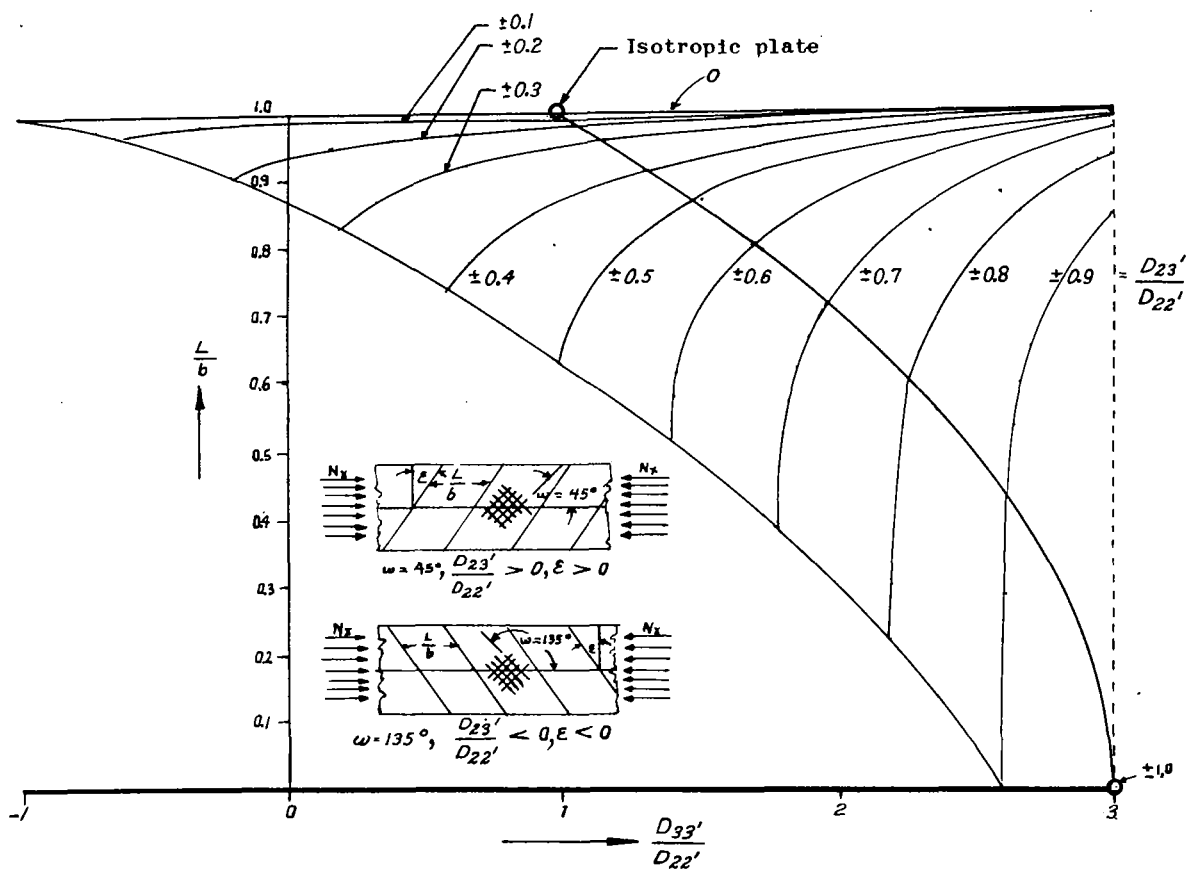


Figure 27.- Half-wave lengths of an orthotropic plate strip under axial forces ($\omega = 45^\circ$ and 135°) (approximate values for free support).

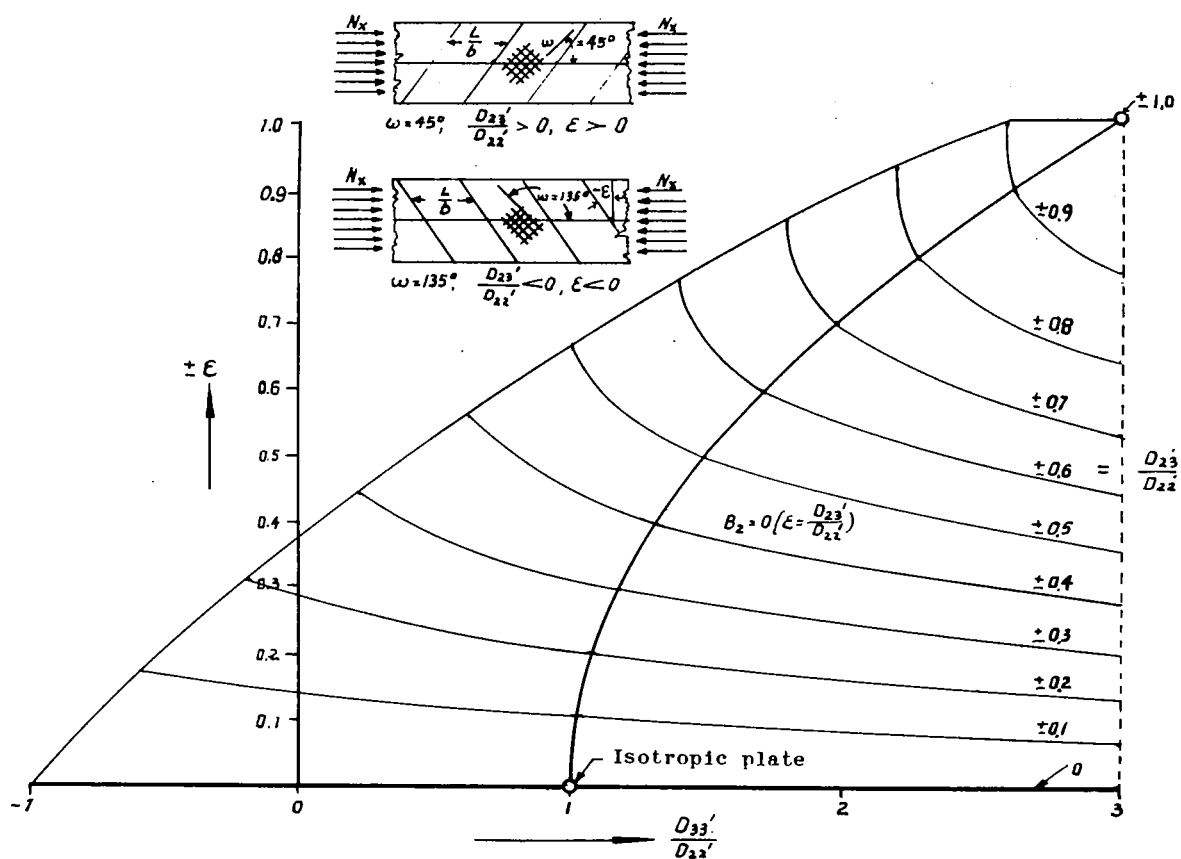


Figure 28.- Inclination of nodal lines of an orthotropic plate under axial force ($\omega = 45^\circ$ and 135°) (approximate values for free support).

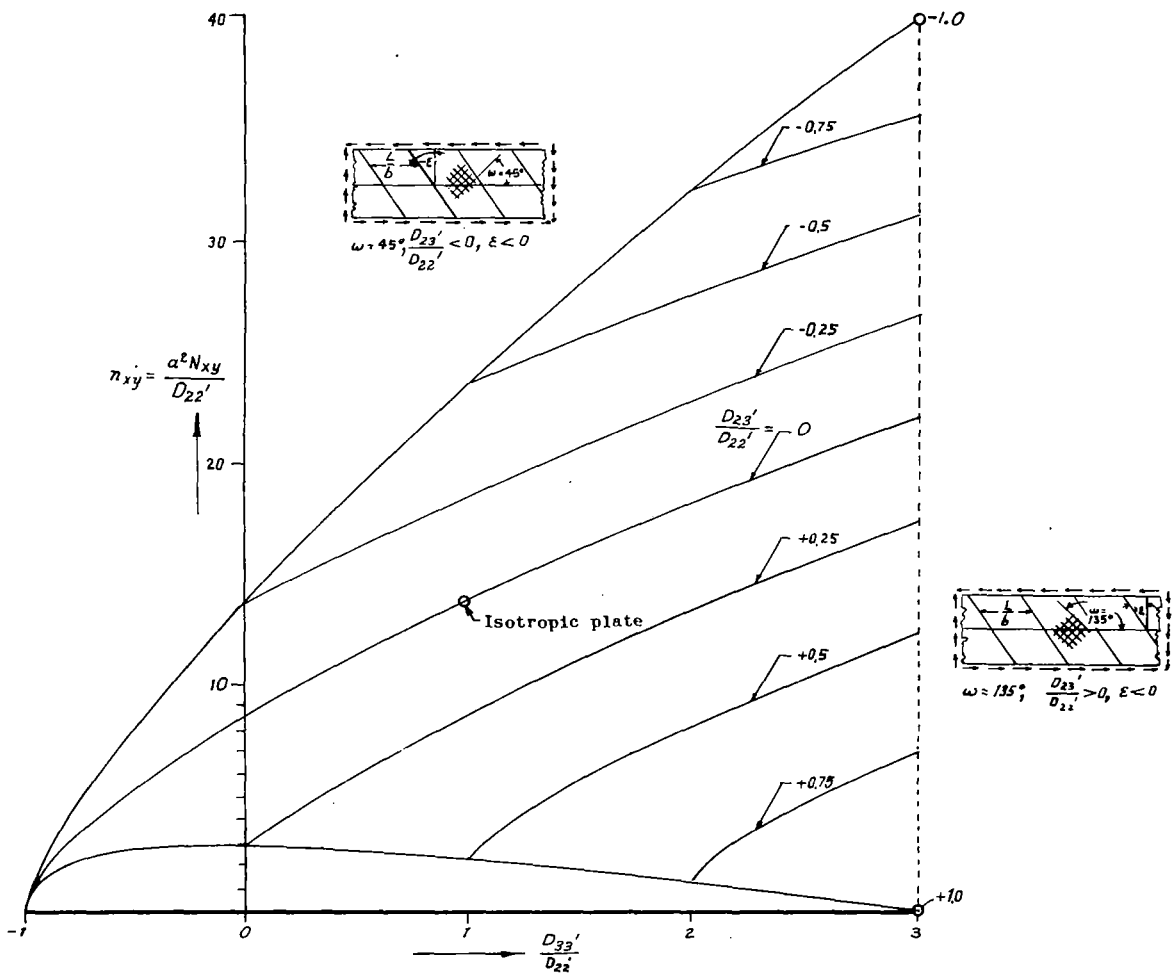


Figure 29.- Buckling load of orthotropic plate strip loaded in shear ($\omega = 45^\circ$ and 135°) (approximate values for free support).

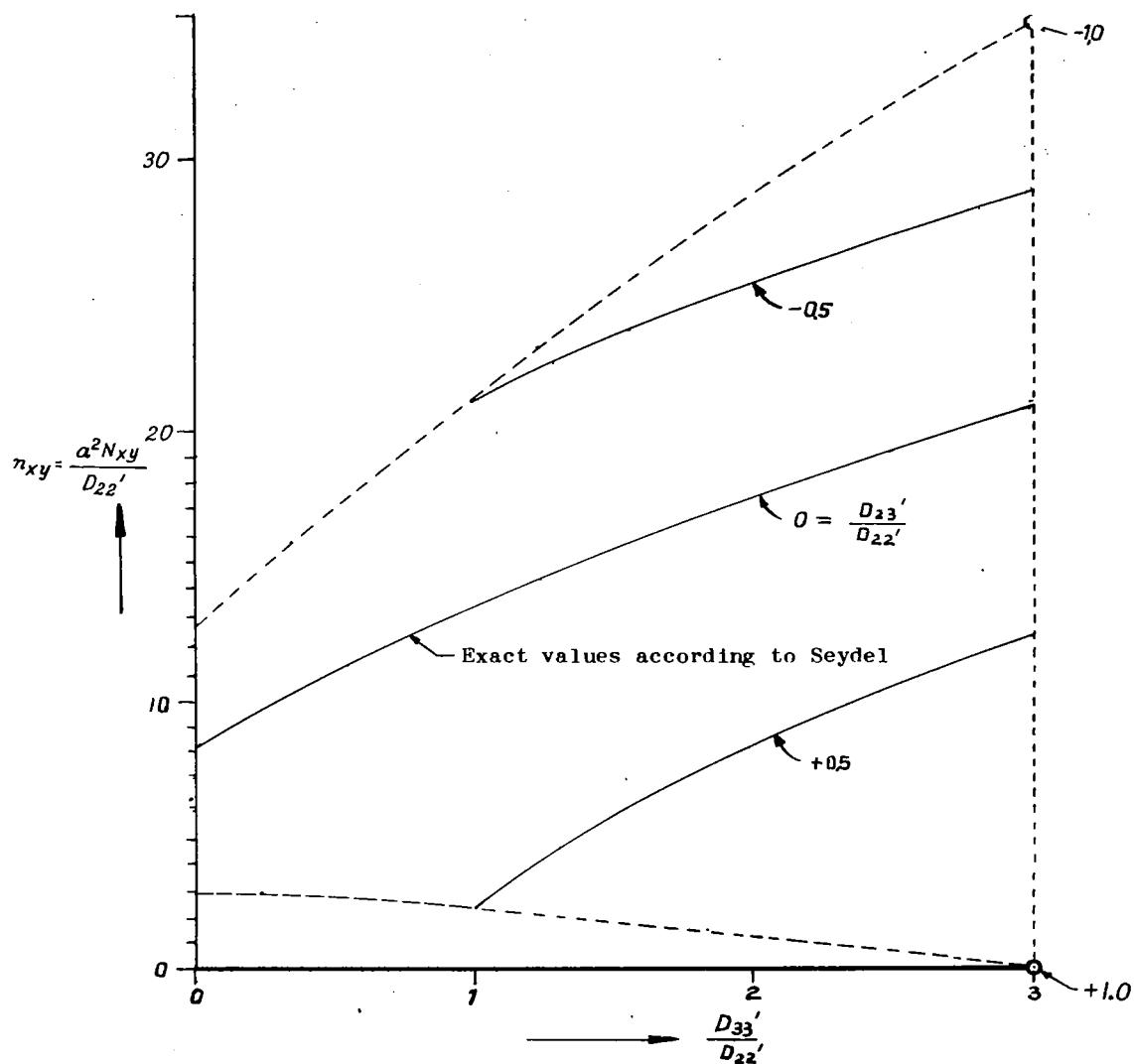


Figure 29(a).- Buckling load of orthotropic plate strip under shear load ($\omega = 45^\circ$ and 135°) - exact values for free support.

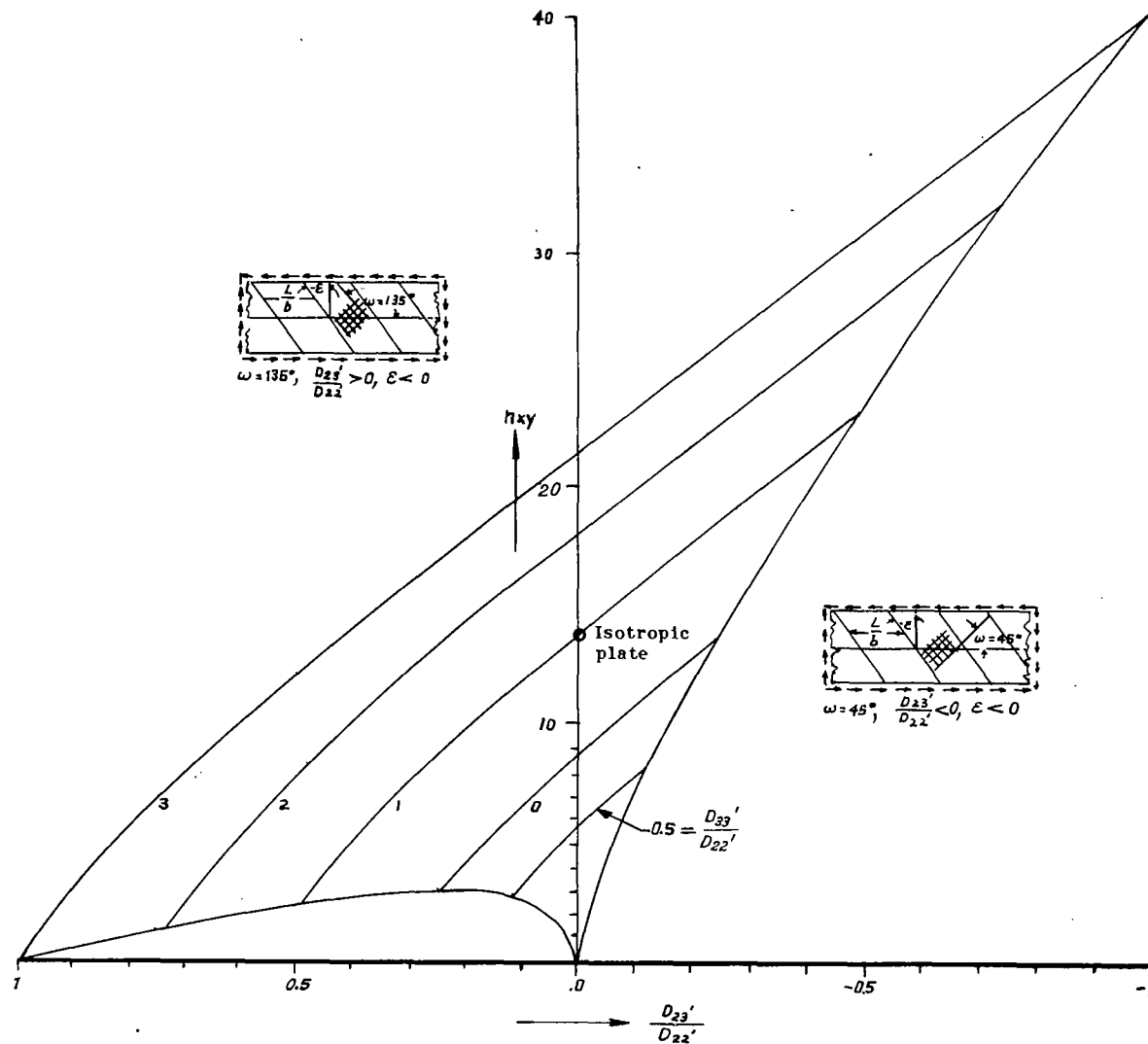
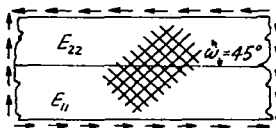
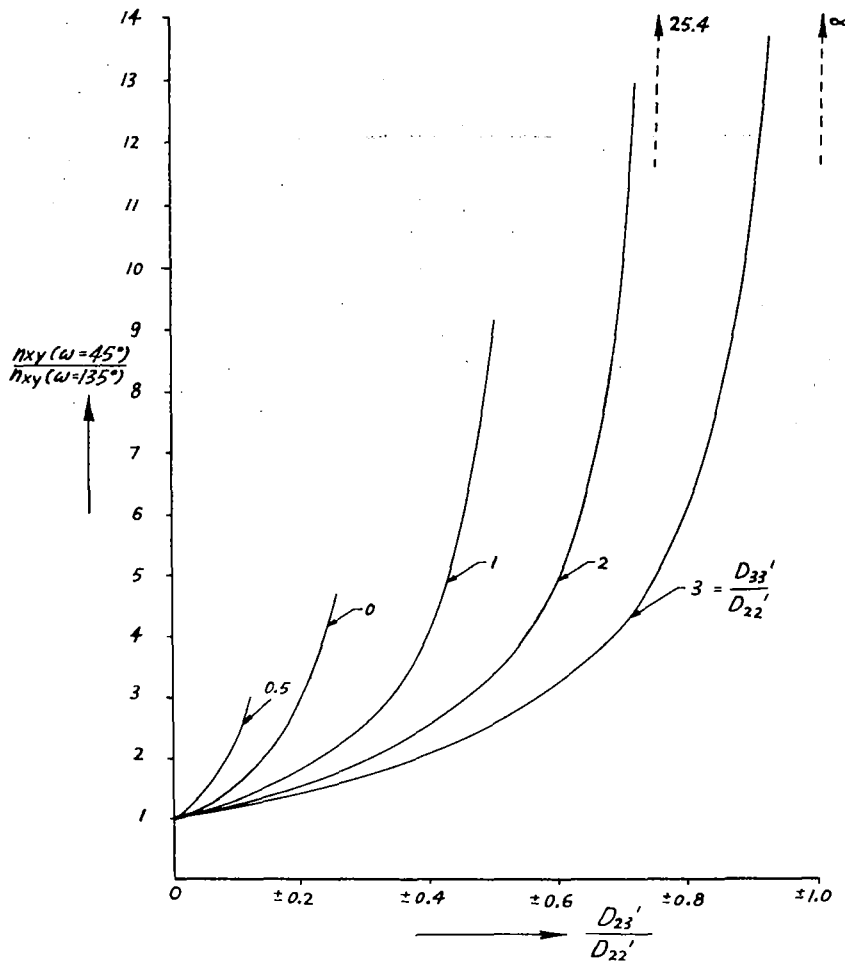
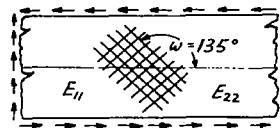


Figure 30.- Buckling load of orthotropic plate strip under shearing force ($\omega = 45^\circ$ and 135°) (approximate values for free support).

Case 1 ($\omega = 45^\circ$)

$$E_{11} > E_{22} : \frac{D_{23}'}{D_{22}'} < 0$$

High buckling strength

Case 2 ($\omega = 135^\circ$)

$$E_{11} < E_{22} : \frac{D_{23}'}{D_{22}'} > 0$$

Low buckling strength

Figure 31.- Relationship between the buckling load of an orthotropic plate strip ($\omega = 45^\circ$ and 135°) and the position of the direction of the maximum flexural stiffness to the direction of the shearing forces (approximate values for free support).

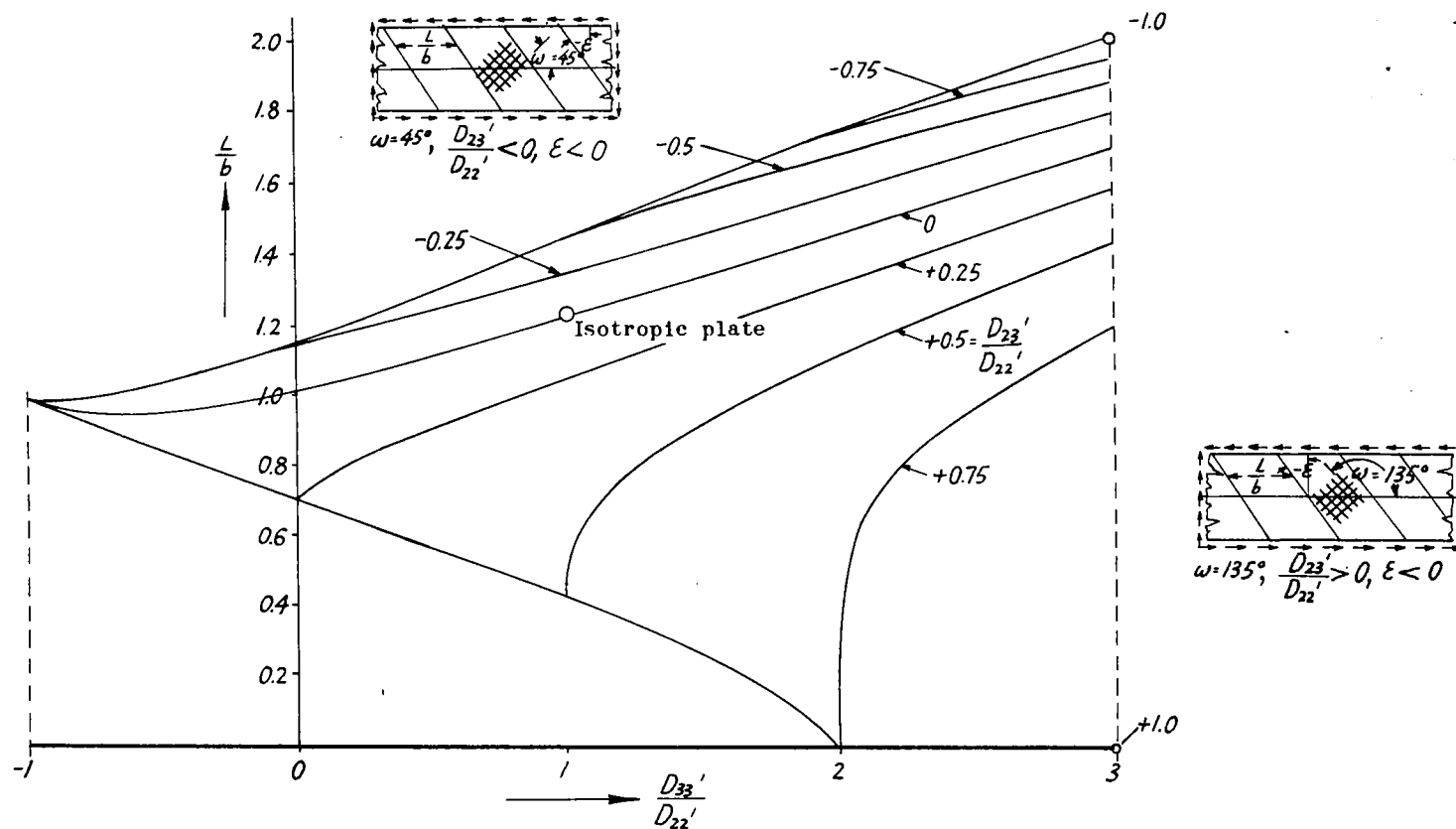


Figure 32.- Half-wave lengths of an orthotropic plate strip loaded in shear ($\omega = 45^\circ$ and 135°) (approximate values for free support).

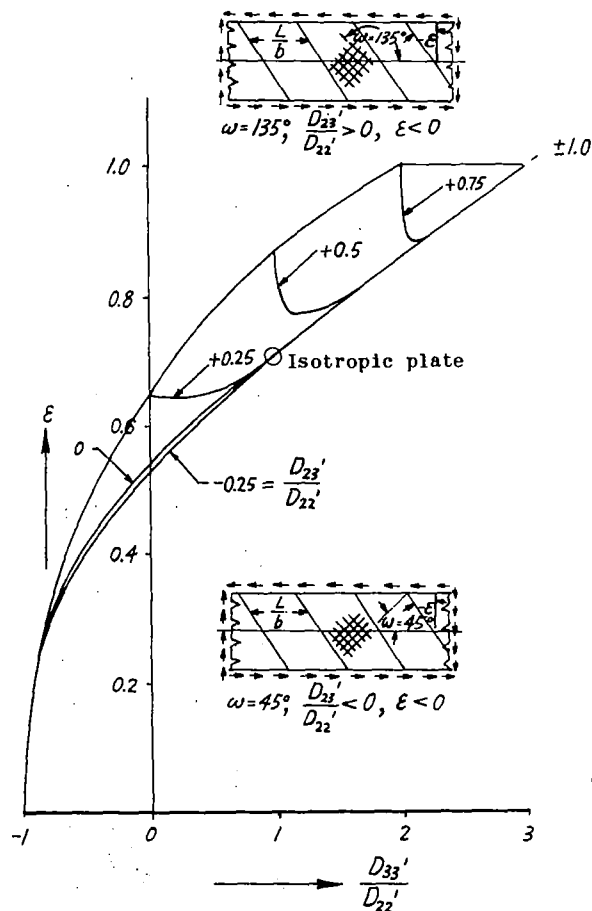


Figure 33.- Inclination of nodal lines of an orthotropic plate strip loaded in shear ($\omega = 45^\circ$ and 135°) (approximate values for free support).

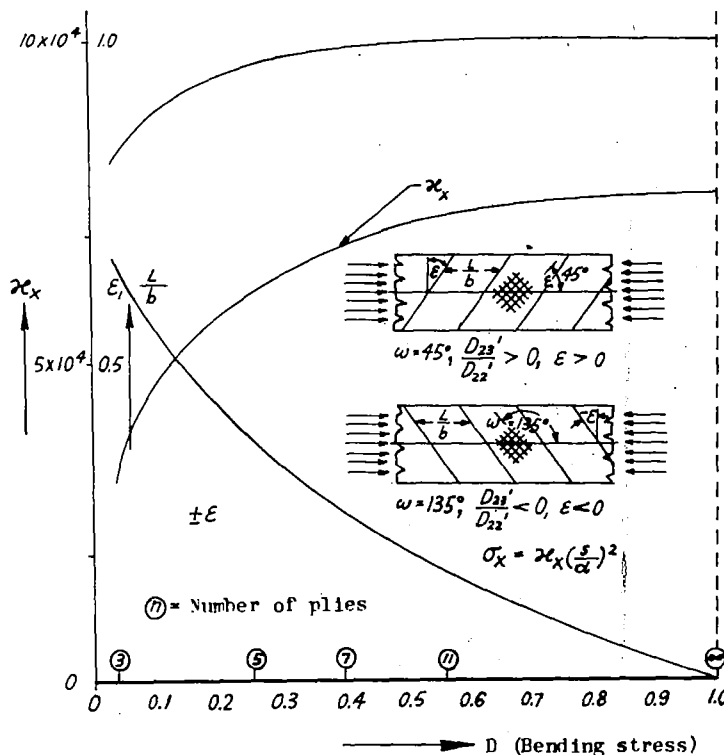


Figure 34.- Buckling stress, buckling length, and nodal line slope of a freely supported plywood strip loaded in compression ($\omega = 45^\circ$ and 135°) plotted against the construction of the plywood (approximate values).

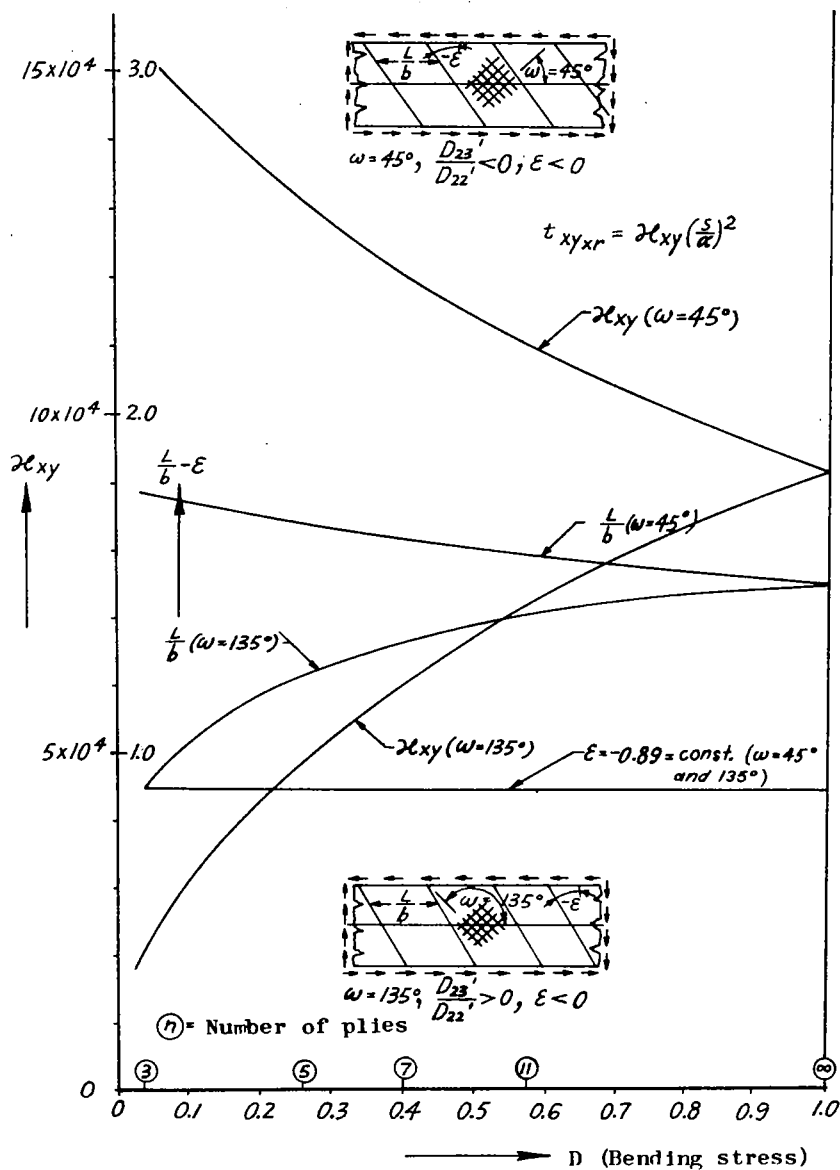


Figure 35.- Buckling stress, buckling length, and nodal line slope of a freely supported plywood strip loaded in shear ($\omega = 45^\circ$ and 135°) plotted against the construction of the plywood.

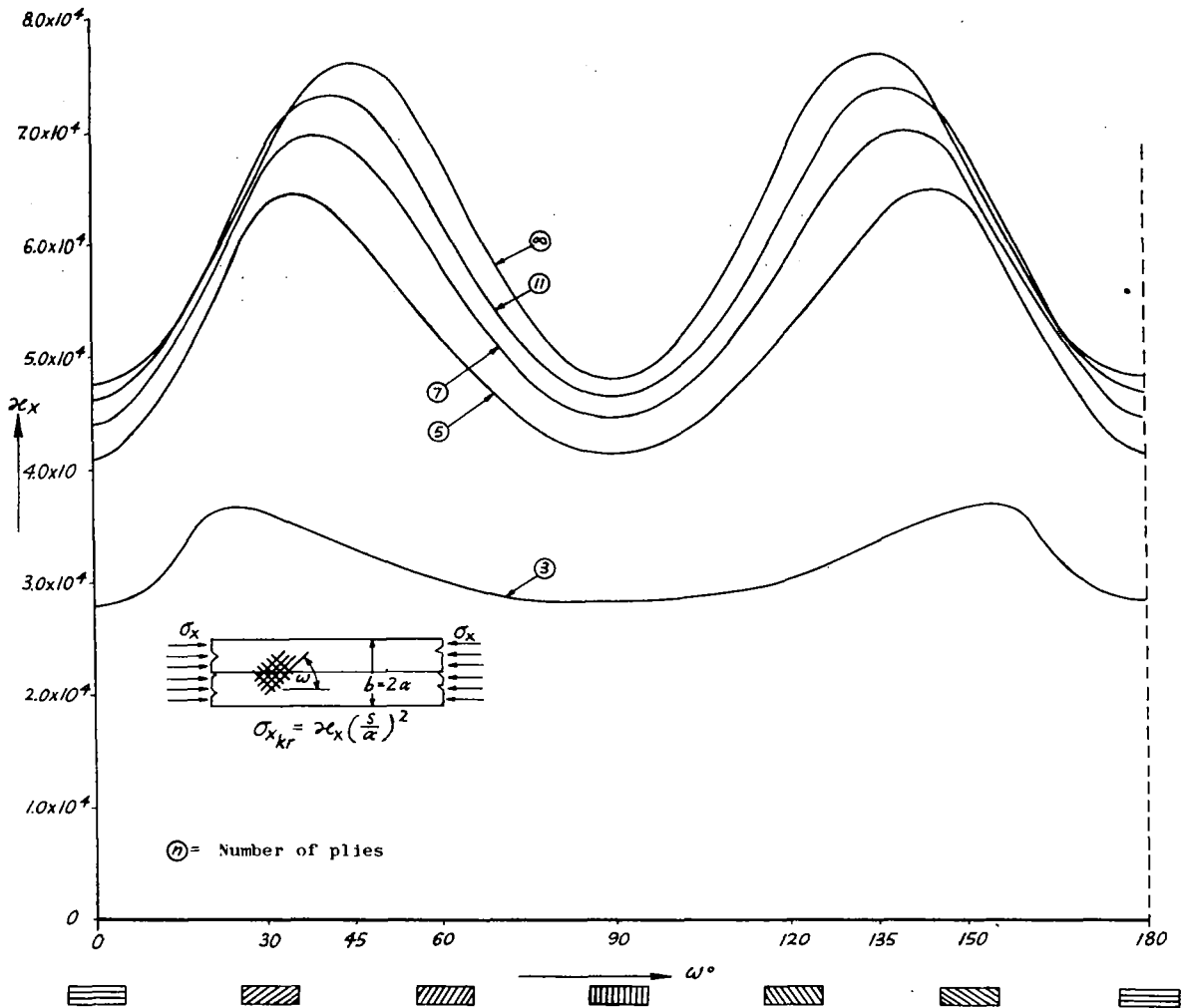


Figure 36.- Buckling stress of freely supported plywood strip loaded in compression plotted against construction of plywood and ω (approximate values).

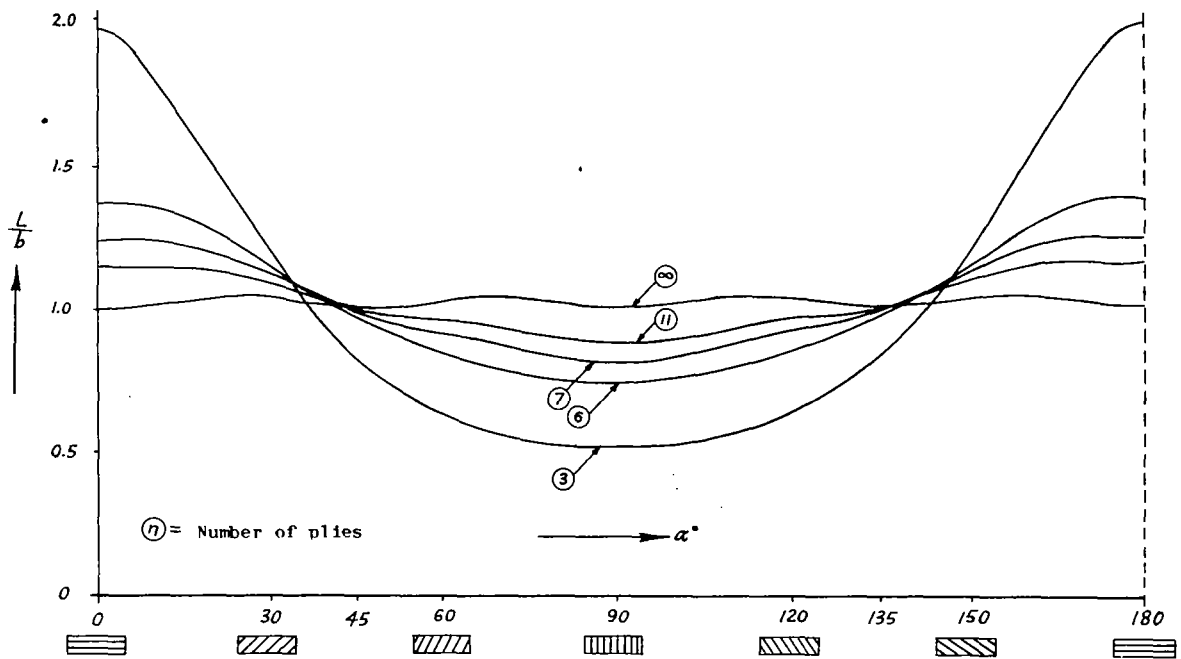


Figure 37.- Buckling length of freely supported plywood strip loaded in compression plotted against construction of plywood and ω (approximate values).

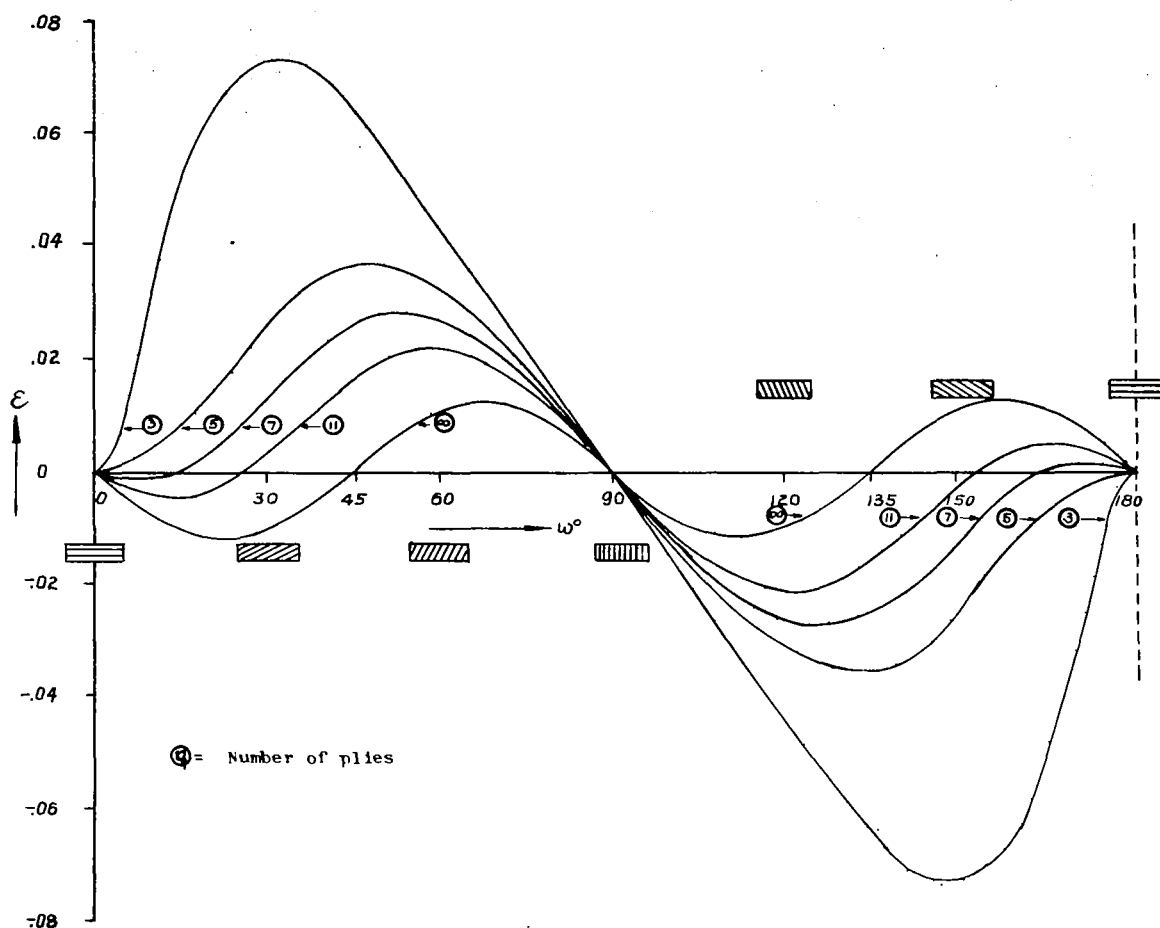


Figure 38.- Nodal line slope of a freely supported plywood strip under compressive load plotted against construction of plywood and ω° (approximate values).

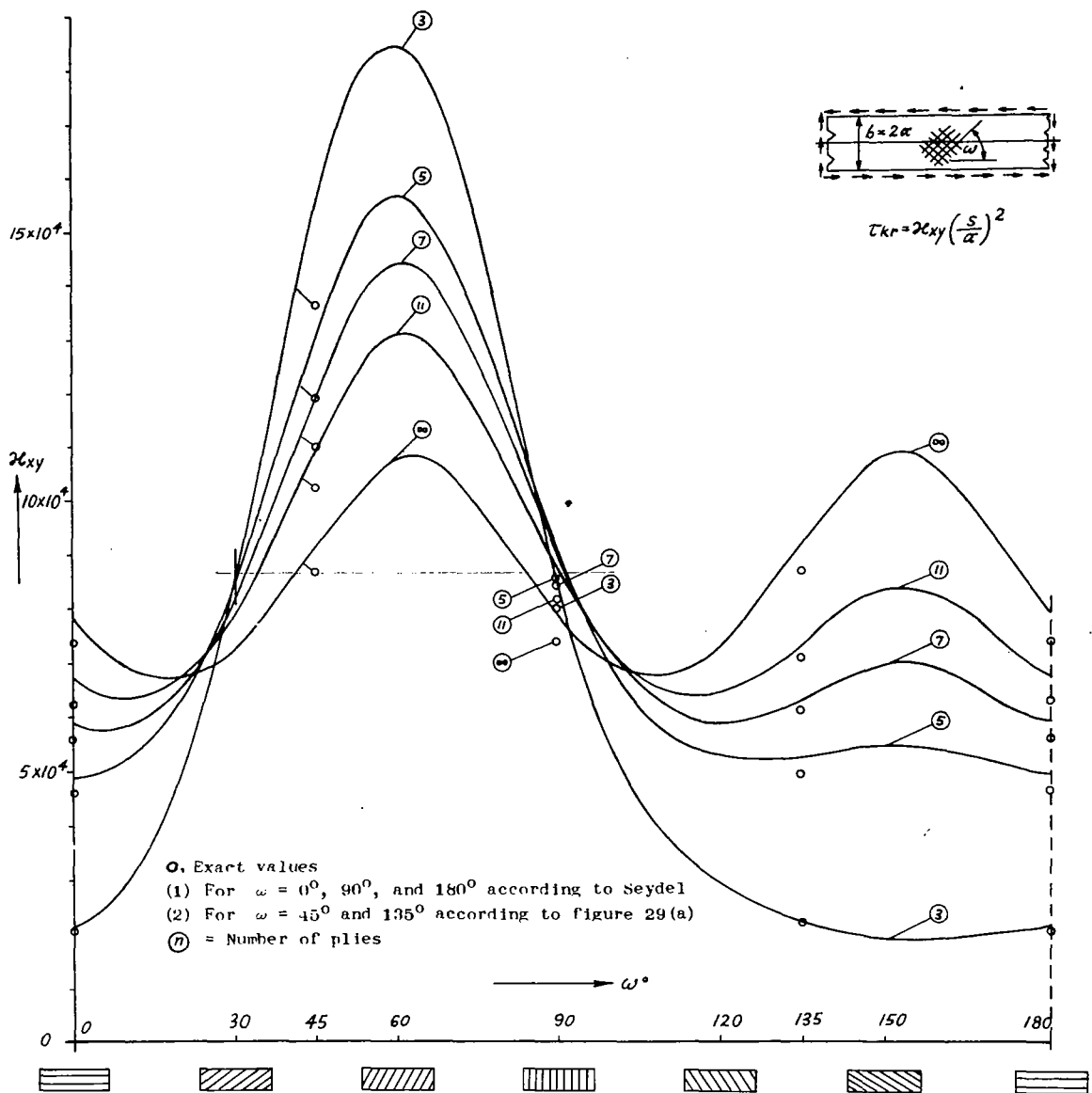


Figure 39.- Buckling stress of freely supported plywood strip under shear load plotted against construction of plywood and ω° (approximate values).

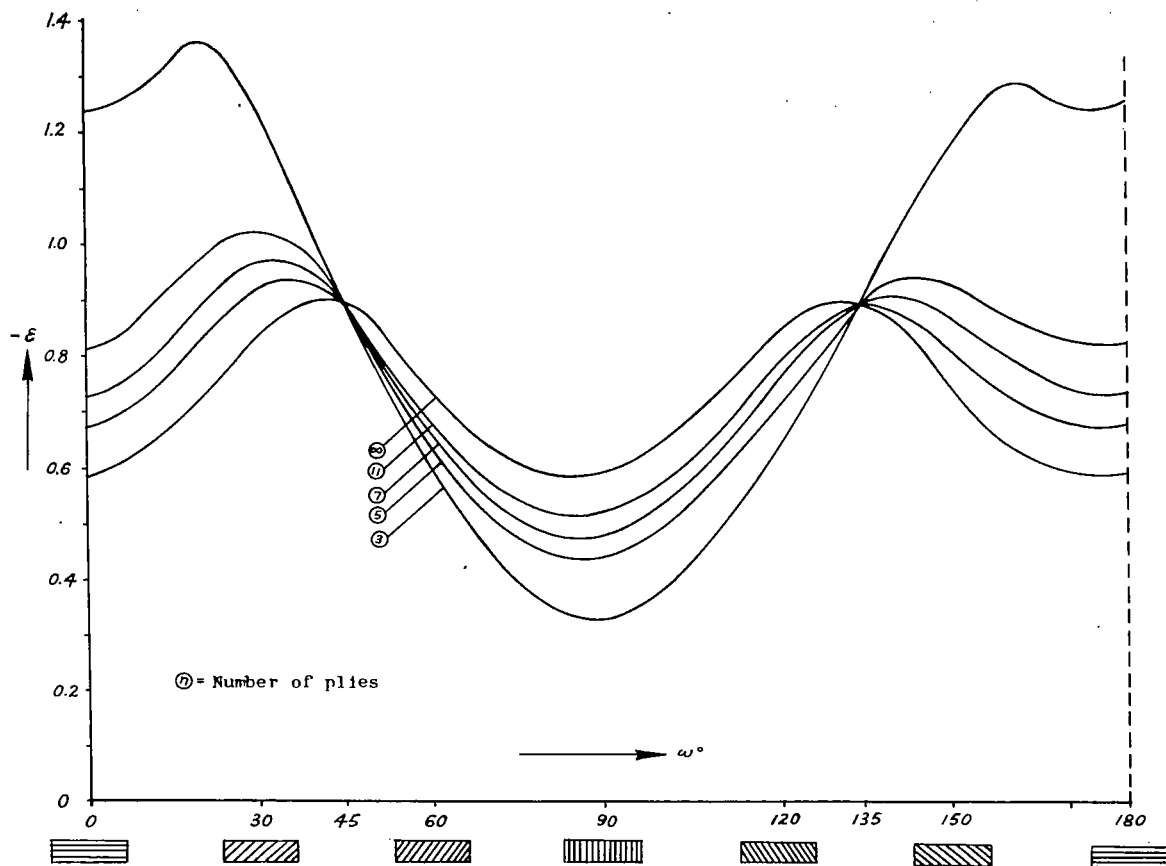


Figure 41.- Nodal line slope of freely supported plywood strip loaded in shear plotted against plywood construction and angle ω (approximate values).

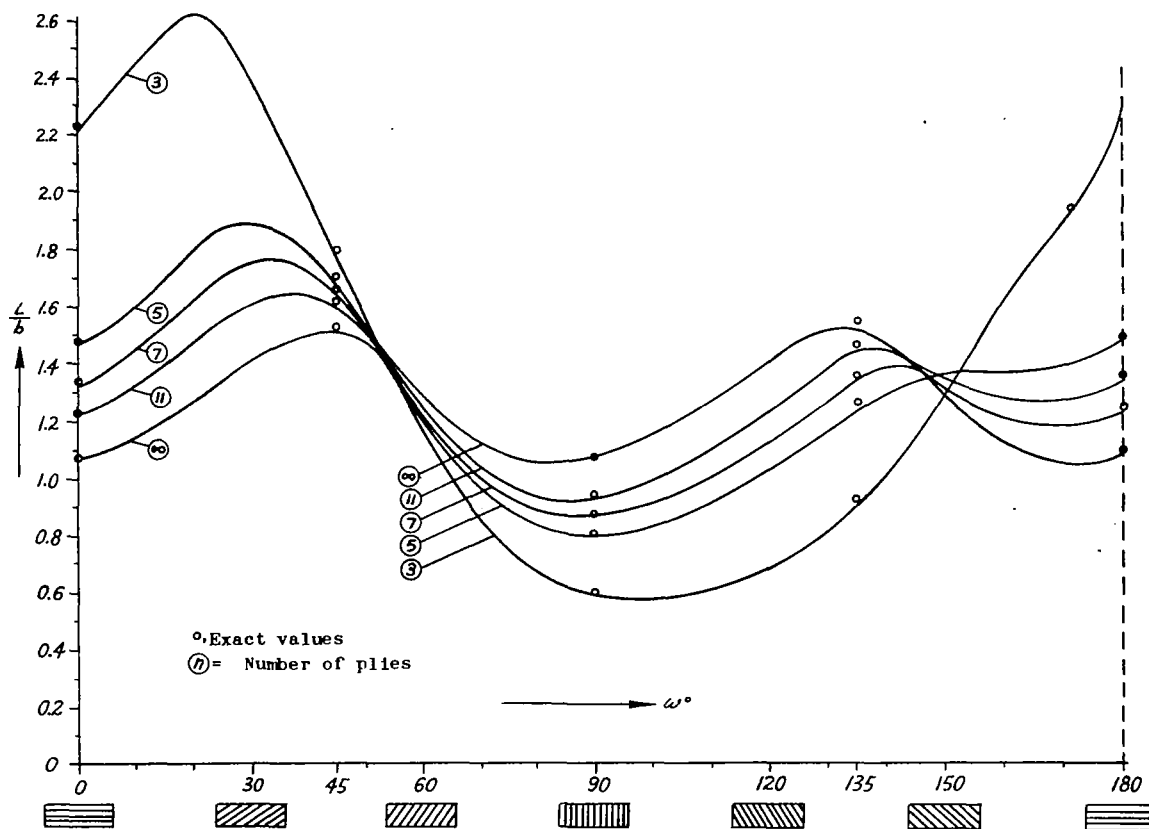


Figure 40.- Buckling length of freely supported plywood strip loaded in shear plotted against construction of plywood and ω° (approximate values).